Series 11 - December 4, 2024

Exercise 1.

Let $\{X_n\}_{n\geqslant 0}$ be the approximation of SDE

$$\begin{aligned} dX(t) &= \lambda X(t)dt + \mu X(t)dW_t, \quad t \in [0,T], \\ X(0) &= X_0, \end{aligned} \tag{1.1}$$

obtained employing the θ-method with time step Δt. Consider $T = 500$, $X_0 = 1$, $\lambda = -1.1$, $\mu = 1$. For $\theta = 0, 1/2, 1$, simulate $\mathbb{E}[X_n^2]$ for $\Delta t = 2$ and comment the results.

Exercise 2.

Study the mean-square and asymptotic stability of the simplified weak θ-scheme:

$$Y_{n+1} = Y_n + \theta\mu Y_{n+1}\Delta t + (1-\theta)\mu Y_n \Delta t + \sigma Y_n \Delta\hat{W}_n \tag{2.1}$$

with $\mathbb{P}(\Delta\hat{W}_n = \pm\sqrt{\Delta t}) = \frac{1}{2}$.

Exercise 3.

Consider the two-dimensional stochastic differential equation:

$$dX_t = b(X_t)dt + \sigma(X_t)dB_t, \quad X_0 = x_0 \tag{3.1}$$

where B_t is a one-dimensional Brownian motion, and

$$b(x) = \begin{pmatrix} x_2 \cos x_1 \\ 2x_1 \sin x_2 \end{pmatrix}, \quad \sigma(x) = \begin{pmatrix} 3 & -0.3 \\ -0.3 & 3 \end{pmatrix} x.$$

1) Show that b and σ satisfy a local Lipschitz condition as well as a monotonicity condition : $\exists \beta_1, \beta_2 \geqslant 0, \quad x^\top b(x) \leqslant \beta_1 |x|^2,\ |\sigma(x)|^2 \leqslant \beta_2 |x|^2$. Deduce that this has a unique solution in L^p for all $p \geqslant 2$.

2) Show that the solution is exponentially asymptotically stable.

   Hint: Consider the Lyapunov function $V(x) = |x|^2 = x_1^2 + x_2^2$ and use the stability condition from Theorem 1 of Lecture 10.

Exercise 4.

Consider the linear SDE in $\mathbb{R}^d$:

$$dX_t = FX_t dt + GX_t dB_t, \quad t > 0,\ X_0 = x_0 \tag{4.1}$$

where $F, G \in \mathbb{R}^{d\times d}$ are commuting, diagonalizable matrices (i.e., they are simultaneously diagonalizable, i.e. $\exists V \in \mathbb{R}^{d\times d}$ invertible s.t. $F = VD_FV^{-1}$, $G = VD_GV^{-1}$, with D_F, D_G diagonal).

1) Show that the solution of the solution of (4.1) is given by:

$$X_t = \exp\Big((F - \frac{1}{2}G^2)t + GB_t\Big)x_0$$

Hint. You can verify it directly using Itô's formula. Use the fact that F and G commute.

2) Assume that all the eigenvalues of $F - \frac{1}{2}G^2$ have negative real parts. Show that the solution is exponentially asymptotically stable.

   Hint: Make a change of variables $Y_t = V^{-1}X_t$ and show asymptotic stability for Y_t.

3) Find the condition on F and G under which the solution is exponentially mean square stable.

4) Analyze the mean square stability of the stochastic θ-scheme for the SDE (4.1). Use complex numbers for the eigenvalues.