Series 10 - November 27, 2024

Exercise 1.

(Implementation of the Milstein–Talay scheme and weak order). Let $f, g\colon \mathbb{R} \to \mathbb{R}$ satisfy the assumptions for existence and uniqueness of the solution of an SDE and consider the following SDE on $[0, T]$

$$\mathrm{d}X(t) = f(X(t))\mathrm{d}t + g(X(t))\mathrm{d}W(t),$$

with initial condition $X(0) = X_0$. Then, consider the scheme

$$\begin{aligned} X_{n+1} = X_n &+ f(X_n)h + g(X_n)\Delta W_n + \frac{1}{2}g'(X_n)g(X_n)I_{1,1} \\ &+ (f'(X_n)f(X_n) + \frac{1}{2}f''(X_n)g^2(X_n))\frac{h^2}{2} + f'(X_n)g(X_n)I_{1,0} \\ &+ (g'(X_n)f(X_n) + \frac{1}{2}g''(X_n)g^2(X_n))I_{0,1}, \end{aligned} \tag{1.1}$$

where

$$I_{0,1} = \int_{t_n}^{t_{n+1}}\int_{t_n}^{s_1} \mathrm{d}s_2\mathrm{d}W(s_1), \qquad I_{1,0} = \int_{t_n}^{t_{n+1}}\int_{t_n}^{s_1} \mathrm{d}W(s_2)\mathrm{d}s_1, \text{and} \qquad I_{1,1} = \int_{t_n}^{t_{n+1}}\int_{t_n}^{s_1} \mathrm{d}W(s_2)\mathrm{d}W(s_1). \tag{1.2}$$

Let $\{\xi_n\}_{n\geqslant 0}$ be a sequence of i.i.d. discrete random variables defined by

$$P(\xi_n = -\sqrt{3}) = \frac{1}{6}, \qquad P(\xi_n = 0) = \frac{2}{3}, \qquad P(\xi_n = +\sqrt{3}) = \frac{1}{6}.$$

Denoting $\chi_n = \xi_n h^{1/2}$, the derivative-free scheme from (1.1) is defined as

$$\begin{aligned} Z_1 &= X_n + f(X_n)h + g(X_n)\chi_n, \\ Z_2^{\pm} &= X_n + f(X_n)h \pm g(X_n)\sqrt{h}, \\ X_{n+1} &= X_n + \frac{1}{2}(f(Z_1) + f(X_n))h + \frac{1}{4}(g(Z_2^+) + g(Z_2^-) + 2g(X_n))\chi_n \\ &\quad + \frac{1}{4}(g(Z_2^+) - g(Z_2^-))(\chi_n^2 - h)\frac{1}{\sqrt{h}}. \end{aligned} \tag{1.3}$$

We remark that under additional assumptions on f and g it can be shown that the above schemes have weak order 2. Set now $f(x) = \lambda x$ with $\lambda = 2$, $g(x) = \mu x$ with $\mu = 0.1$, $X_0 = 1$ and $T = 1$. Verify numerically that the schemes (1.1) (computed considering Brownian increments $(\Delta W_n)_n$) and (1.3) (computed considering r.v.s $(\chi_n)_n$) have weak order 2. Choose different step sizes $h = 2^{-i}$ with $i = 1, \dots, 5$ and approximate the expectations using $M = 10^4$ realizations of $\{\xi_n\}_{n\geqslant 0}$.

Solution

The plot is given in Figure 1.

Exercise 2.

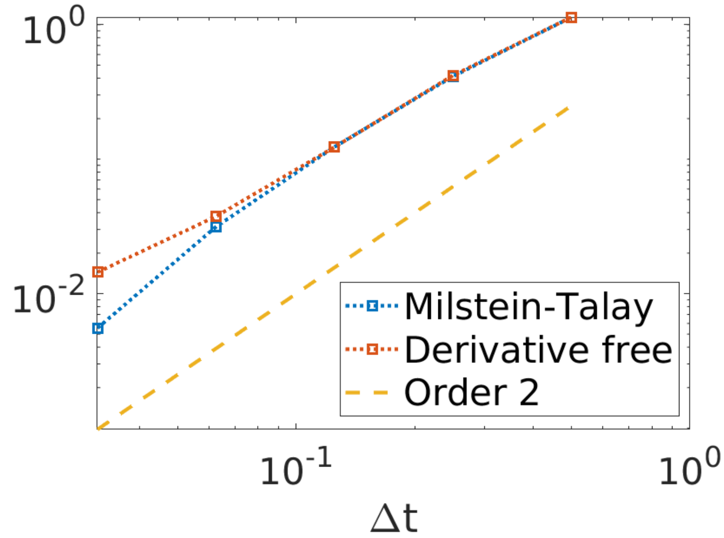

Figure 1: Weak order of convergence of the Milstein–Talay scheme and its derivative free version.

For an SDE in $[0, T]$ of the form

$$\begin{aligned} dX(t) &= f(X(t))dt + g(X(t))dW(t), \\ X(0) &= X_0, \end{aligned} \tag{2.1}$$

consider the Euler–Maruyama method

$$X_n = X_{n-1} + f(X_{n-1})\Delta t + g(X_{n-1})(W(t_n) - W(t_{n-1})), \tag{2.2}$$

and the Milstein scheme

$$\begin{aligned} X_n = X_{n-1} + f(X_{n-1})\Delta t + g(X_{n-1})(W(t_n) - W(t_{n-1})) \\ + \tfrac{1}{2}g'(X_{n-1})g(X_{n-1})((W(t_n) - W(t_{n-1}))^2 - \Delta t). \end{aligned}$$

Apply the two schemes to the SDE (2.1) setting the final time $T = 1$, the initial condition $X_0 = 1$ and the functions $f(x) = \lambda x$, $g(x) = \mu x$ with $\lambda = 2$, $\mu = 1$. Compute numerically the strong and weak orders of convergence of the methods. Choose different step sizes $\Delta t = 2^{-i}$ with $i = 5, \dots, 10$ and approximate the expectations using $M = 10^4$ realizations of the Brownian motion.

Solution

The plots are given in Figure 2.

Exercise 3.

Consider the SDE

$$dX_t = a(t, X_t)dt + b(t, X_t)dW_t, \quad X(0) = X_0, \tag{3.1}$$

where $a : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $b : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are Lipschitz in the spatial variable, uniformly in t, with constant L and satisfy the linear-growth bound property.

Let $\{Y_n\}_{n\geqslant 0}$ be the approximation of (3.1) defined by the following method

$$Y_{n+1} = Y_n + \hat{a}(t_{n+1}, Y_{n+1})\Delta t + b(t_{n+1}, Y_{n+1})\xi_n \tag{3.2}$$

where $\mathbb{P}(\xi_n = \pm\sqrt{\Delta t}) = \frac{1}{2}$ and $\hat{a} = a - bb'$.

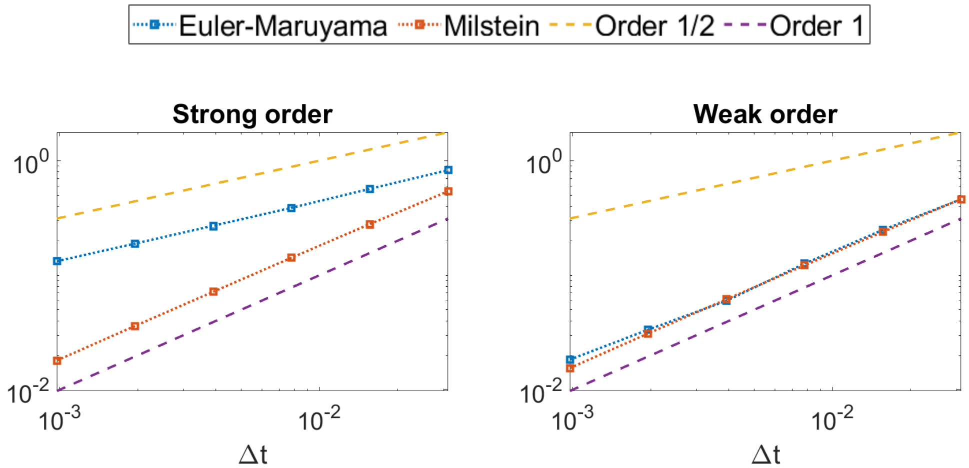

Figure 2: Strong and weak orders of convergence for the Euler–Maruyama and Milstein schemes.

1) Show that there exists a unique solution of (3.2) for $\Delta t < \frac{1}{L^2}$.
2) Show numerically that (3.2) is converging to (3.1) weakly, while it is not the case for the method (3.2) with $\hat{a}$ is replaced by a. To do so, you can apply (3.2) a geometric Brownian motion with a scalar $\mu = -1$ in the drift, a scalar $\sigma = 0.3$ in the diffusion, $X_0 = 1$ and approximate the average with $M = 10^5$ realizations.
3) Assume that (3.2) and (3.1) have bounded p-moments with $p \geqslant 2$. Show that (3.2) is actually convergent of weak order 1.

Solution

1) Consider that ξ_n can assume just two values, i.e. $\pm\sqrt{\Delta t}$. Suppose $\xi_n = \sqrt{\Delta t}$ and define $\varphi(x) := Y_n + \hat{a}(t_{n+1}, Y_{n+1})\Delta t + b(t_{n+1}, x)\sqrt{\Delta t}$. Then
$$|\varphi(x) - \varphi(y)| \leqslant \sqrt{\Delta t}|b(t_{n+1}, x) - b(t_{n+1}, y)| \leqslant L\sqrt{\Delta t}|x - y|.$$
Taking $\Delta t < \frac{1}{L^2}$, φ is a contractive map and, via Banach Theorem, there exists a unique fixed point y of φ, i.e. $y = \varphi(y)$. The case $\xi_n = -\sqrt{\Delta t}$ is analogous. Showing that the approximation is unique for all the possible (two) realizations yields the thesis.
2) From Figure (3) one can notice that the modified-drift implicit method is weakly convergent.

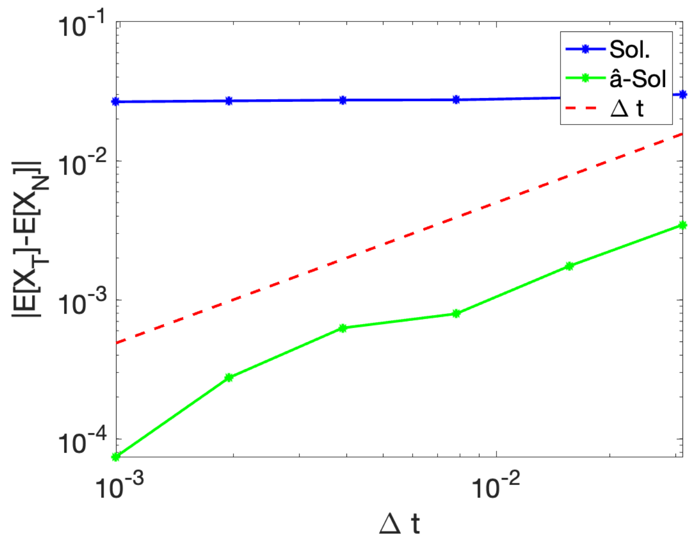

Figure 3: Weak convergence for the fully implicite method without the modified drift (blue) and with the modified drift (green) applied to a geometric Brownian motion.