Series Break - October 23, 2024

Exercise 1.

Let $(W(t), t \geqslant 0)$ be a standard one-dimensional Brownian motion. The goal of this exercise is to show that for the class of Hermite polynomials $\{h_n\}_{n\in\mathbb{N}}$ it holds

$$\int_0^t h_n(W(s); s)\mathrm{d}W(s) = \frac{h_{n+1}(W(t); t)}{n+1}. \tag{1.1}$$

Remark. The function $h_n(W(t); t)$ plays the role that t^n plays in ordinary calculus.
For $n \in \mathbb{N}$, define the n-th Hermite polynomial as

$$h_n(x; \rho) = (-\rho)^n e^{\frac{x^2}{2\rho}} \frac{\mathrm{d}^n}{\mathrm{d}x^n}(e^{-\frac{x^2}{2\rho}}),$$

where ρ is a parameter.

i) Compute $h_n(x; \rho)$ for $n = 0, \dots, 4$.

ii) Show that

$$e^{tx-\frac{t^2\rho}{2}} = \sum_{n=0}^{\infty} \frac{h_n(x; \rho)}{n!} t^n.$$

iii) Show that

$$e^{\rho ts} = \sum_{n,m=0}^{\infty} \frac{t^n s^m}{n!m!} \int_{\mathbb{R}} h_n(x; \rho)h_m(x; \rho)\nu(x; \rho)\mathrm{d}x.,$$

where

$$\nu(x; \rho) = \frac{1}{\sqrt{2\pi\rho}} e^{-\frac{x^2}{2\rho}}.$$

iv) Show that the Hermite polynomials $\{h_n(x; \rho)\}_{n\geqslant 0}$ are orthogonal in $\mathrm{L}^2(\mathbb{R})$ with respect to the weight function $\nu(x; \rho)$ and that

$$\int_{\mathbb{R}} h_n(x; \rho)^2 \nu(x; \rho)\mathrm{d}x = n!\rho^n.$$

v) Using the fact that the stochastic process $Y(t) = e^{\lambda W(t) - \frac{\lambda^2 t}{2}}$ satisfies

$$Y(t) = Y(0) + \lambda \int_0^t Y(s)\,\mathrm{d}W(s), \tag{1.2}$$

prove equality (1.1).
Remark. Equation (1.2) will be proved in a future series of exercises.

Exercise 2.

(Itô product rule) Consider for $0 \leqslant t \leqslant T$ the stochastic differentials

$$dX_1 = F_1\mathrm{d}t + G_1\mathrm{d}W,$$
$$dX_2 = F_2\mathrm{d}t + G_2\mathrm{d}W,$$

where $F_1, F_2 \in \mathrm{M}^1(0,T)$ and $G_1, G_2 \in \mathrm{M}^2(0,T)$. Assume that $X_1(0) = X_2(0) = 0$ and let F_1, F_2, G_1, G_2 be time independent $\mathcal{F}(0)$-measurable random variables. Without using the Itô formula, show that

$$\mathrm{d}(X_1 X_2) = X_2 \mathrm{d}X_1 + X_1 \mathrm{d}X_2 + G_1 G_2 \mathrm{d}t.$$

Exercise 3.

Let $(W(t), t \geqslant 0)$ be a one-dimensional Brownian motion. Using the Itô formula:

i) compute $\mathrm{d}(W^m)$ where m is a positive integer,

ii) show that $Y(t) = e^{\lambda W(t) - \frac{\lambda^2 t}{2}}$ where $\lambda \in \mathbb{R}$ satisfies

$$\begin{cases} \mathrm{d}Y = \lambda Y \mathrm{d}W, \\ Y(0) = 1. \end{cases}$$

Exercise 4.

Let $(X(t), t \geqslant 0)$ be an n-dimensional process with stochastic differential

$$\mathrm{d}X(t) = F(t)\mathrm{d}t + G(t)\mathrm{d}W(t),$$

where $F(t) \in \mathbb{R}^n$, $G(t) \in \mathbb{R}^{n\times m}$ and $W(t)$ is an m-dimensional Brownian motion. Consider $u(x) \colon \mathbb{R}^n \to \mathbb{R}$ defined as $u(x) = |x|^2 = \sum_{i=1}^n x_i^2$. Using the Itô formula, compute the stochastic differential $\mathrm{d}u(X(t))$ and give its expression in integral form.

Exercise 5.

Let $(W_1(t), t \geqslant 0)$ and $(W_2(t), t \geqslant 0)$ be two independent one-dimensional Brownian motions. Without using the Itô formula, show that

$$\mathrm{d}(W_1 W_2) = W_1 \mathrm{d}W_2 + W_2 \mathrm{d}W_1.$$

Hint. Consider $X(t) = \frac{1}{\sqrt{2}}(W_1(t) + W_2(t))$.

Exercise 6.

Let $B(t)$ be a one-dimensional standard Gaussian process such that $\mathbb{E}[B(t)B(s)] = \min\{s,t\}$. Show that $B(t)$ is a Brownian motion.
Recall. A real-valued stochastic process $(X(t), t \geqslant 0)$ is called a one-dimensional Gaussian process if for any integer $n \geqslant 1$ and any choice of times $t_1 < \ldots < t_n$ the random vector $(X(t_1), \ldots, X(t_n))$ has a multivariate Gaussian distribution. Moreover, it is called a standard one-dimensional Gaussian process if in addition $E[X(t)] = 0$ for all $t \geqslant 0$.