Series Break - October 23, 2024

Exercise 1.

Let $(W(t), t \geqslant 0)$ be a standard one-dimensional Brownian motion. The goal of this exercise is to show that for the class of Hermite polynomials $\{h_n\}_{n\in\mathbb{N}}$ it holds

$$\int_0^t h_n(W(s); s)\mathrm{d}W(s) = \frac{h_{n+1}(W(t); t)}{n+1}. \tag{1.1}$$

Remark. The function $h_n(W(t); t)$ plays the role that t^n plays in ordinary calculus.
For $n \in \mathbb{N}$, define the n-th Hermite polynomial as

$$h_n(x; \rho) = (-\rho)^n e^{\frac{x^2}{2\rho}} \frac{\mathrm{d}^n}{\mathrm{d}x^n}(e^{-\frac{x^2}{2\rho}}),$$

where ρ is a parameter.

i) Compute $h_n(x; \rho)$ for $n = 0, \dots, 4$.

ii) Show that

$$e^{tx - \frac{t^2\rho}{2}} = \sum_{n=0}^{\infty} \frac{h_n(x; \rho)}{n!} t^n.$$

iii) Show that

$$e^{\rho ts} = \sum_{n,m=0}^{\infty} \frac{t^n s^m}{n!m!} \int_{\mathbb{R}} h_n(x; \rho) h_m(x; \rho) \nu(x; \rho) \mathrm{d}x.,$$

where

$$\nu(x; \rho) = \frac{1}{\sqrt{2\pi\rho}} e^{-\frac{x^2}{2\rho}}.$$

iv) Show that the Hermite polynomials $\{h_n(x; \rho)\}_{n\geqslant 0}$ are orthogonal in $\mathrm{L}^2(\mathbb{R})$ with respect to the weight function $\nu(x; \rho)$ and that

$$\int_{\mathbb{R}} h_n(x; \rho)^2 \nu(x; \rho) \mathrm{d}x = n!\rho^n.$$

v) Using the fact that the stochastic process $Y(t) = e^{\lambda W(t) - \frac{\lambda^2 t}{2}}$ satisfies

$$Y(t) = Y(0) + \lambda \int_0^t Y(s)\,\mathrm{d}W(s), \tag{1.2}$$

prove equality (1.1).
Remark. Equation (1.2) will be proved in a future series of exercises.

Solution

i)

$$h_0(x; \rho) = 1, \quad h_1(x; \rho) = x, \quad h_2(x; \rho) = x^2 - \rho,$$
$$h_3(x; \rho) = x^3 - 3\rho x, \quad h_4(x; \rho) = x^4 - 6\rho x^2 + 3\rho^2.$$

ii) First, rewrite $e^{tx-\frac{t^2\rho}{2}} = e^{\frac{x^2}{2\rho}}e^{-\frac{(x-t\rho)^2}{2\rho}}$. Then, consider the function $f(y) = e^{-\frac{y^2}{2\rho}}$ and write its Taylor expansion around the point $y_0 = x$ and evaluated in $y = x - t\rho$. We obtain

$$e^{-\frac{(x-t\rho)^2}{2\rho}} = \sum_{n=0}^{\infty} \frac{1}{n!}\frac{d^n}{dx^n}\left(e^{-\frac{x^2}{2\rho}}\right)(-t\rho)^n,$$

which implies

$$e^{tx-\frac{t^2\rho}{2}} = \sum_{n=0}^{\infty}(-\rho)^n e^{\frac{x^2}{2\rho}}\frac{\mathrm{d}^n}{\mathrm{d}x^n}(e^{-\frac{x^2}{2\rho}})\frac{t^n}{n!} = \sum_{n=0}^{\infty} h_n(x;\rho)\frac{t^n}{n!}.$$

iii) Using *ii)* we have

$$e^{tx-\frac{t^2\rho}{2}}e^{sx-\frac{s^2\rho}{2}} = \sum_{n,m=0}^{\infty}\frac{t^n s^m}{n!m!}h_n(x;\rho)h_m(x;\rho),$$

and hence

$$\begin{aligned}\sum_{n,m=0}^{\infty}\frac{t^n s^m}{n!m!}\int_{\mathbb{R}} h_n(x;\rho)h_m(x;\rho)\nu(x;\rho)\mathrm{d}x &= \int_{\mathbb{R}} e^{(t+s)x-\frac{\rho(t^2+s^2)}{2}}\nu(x;\rho)\mathrm{d}x \\ &= e^{\rho ts}\int_{\mathbb{R}} e^{(t+s)x-\frac{\rho(t+s)^2}{2}}\nu(x;\rho)\mathrm{d}x = e^{\rho ts}\underbrace{\frac{1}{\sqrt{2\pi\rho}}\int_{\mathbb{R}} e^{-\frac{(x-\rho(t+s))^2}{2\rho}}\mathrm{d}x}_{=1}.\end{aligned}$$

iv) Note that

$$e^{\rho st} = \sum_{n=0}^{\infty}\frac{(\rho ts)^n}{n!} = 1 + \rho ts + \rho^2\frac{(ts)^2}{2!} + \rho^3\frac{(ts)^3}{3!} + \dots,$$

so identifying this with the right hand side of *iii)*, we conclude that for $m \neq n$,

$$\int_{\mathbb{R}} h_n(x;\rho)h_m(x;\rho)\nu(x;\rho)\mathrm{d}x = 0,$$

and

$$\int_{\mathbb{R}} h_n(x;\rho)^2\nu(x;\rho)\mathrm{d}x = n!\rho^n.$$

v) In order to show (1.1), we have on one hand (using the formula in *ii)*)

$$Y(t) = e^{\lambda W(t)-\frac{\lambda^2 t}{2}} = \sum_{n=0}^{\infty}\frac{h_n(W(t);t)}{n!}\lambda^n = 1 + \sum_{n=1}^{\infty}\frac{h_n(W(t);t)}{n!}\lambda^n,$$

and on the other hand (using (1.2))

$$\begin{aligned}Y(t) &= Y(0) + \lambda\int_0^t\sum_{n=0}^{\infty}\frac{h_n(W(s);s)}{n!}\lambda^n\mathrm{d}W(s) \\ &= 1 + \sum_{n=1}^{\infty}\Big(\int_0^t\frac{h_{n-1}(W(s);s)}{(n-1)!}\mathrm{d}W(s)\Big)\lambda^n.\end{aligned}$$

As these equalities are valid for any λ, identifying the factor of the λ^n we obtain the result.

Exercise 2.

(Itô product rule) Consider for $0 \leqslant t \leqslant T$ the stochastic differentials

$$\begin{aligned}dX_1 &= F_1\mathrm{d}t + G_1\mathrm{d}W, \\ dX_2 &= F_2\mathrm{d}t + G_2\mathrm{d}W,\end{aligned}$$

where $F_1, F_2 \in \mathrm{M}^1(0,T)$ and $G_1, G_2 \in \mathrm{M}^2(0,T)$. Assume that $X_1(0) = X_2(0) = 0$ and let F_1, F_2, G_1, G_2 be time independent $\mathcal{F}(0)$-measurable random variables. Without using the Itô formula, show that

$$\mathrm{d}(X_1 X_2) = X_2 \mathrm{d}X_1 + X_1 \mathrm{d}X_2 + G_1 G_2 \mathrm{d}t.$$

Solution

For $i = 1, 2$, we have

$$X_i(t) = X_i(0) + \int_0^t F_i \mathrm{d}t + \int_0^t G_i \mathrm{d}W = F_i t + G_i W(t).$$

Therefore, we obtain

$$\begin{aligned}
&\int_0^r (X_2 \mathrm{d}X_1 + X_1 \mathrm{d}X_2 + G_1 G_2)\mathrm{d}t \\
&\quad = \int_0^r X_2(F_1 \mathrm{d}t + G_1 \mathrm{d}W) + \int_0^r X_1(F_2 \mathrm{d}t + G_2 \mathrm{d}W) + \int_0^r G_1 G_2 \mathrm{d}t \\
&\quad = \int_0^r (X_2 F_1 + X_1 F_2 + G_1 G_2)\mathrm{d}t + \int_0^r (X_2 G_1 + X_1 G_2)\mathrm{d}W \\
&\quad = F_1 \int_0^r (F_2 t + G_2 W(t))\mathrm{d}t + F_2 \int_0^r (F_1 t + G_1 W(t))\mathrm{d}t + \int_0^r G_1 G_2 \mathrm{d}t \\
&\qquad\qquad + G_1 \int_0^r (F_2 t + G_2 W(t))\mathrm{d}W + G_2 \int_0^r (F_1 t + G_1 W(t))\mathrm{d}W \\
&\quad = (F_1 F_2 + F_2 F_1)\frac{r^2}{2} + G_1 G_2 r \\
&\qquad\qquad + (F_1 G_2 + F_2 G_1)\Big(\int_0^r W \mathrm{d}t + \int_0^r t \mathrm{d}W\Big) + G_1 G_2\Big(2\int_0^r W \mathrm{d}W\Big) \\
&\quad = F_1 F_2 r^2 + G_1 G_2 (r + W(r)^2 - r) + (F_1 G_2 + F_2 G_1) r W(r).
\end{aligned}$$

where the last equality follows from the previous exercise. Finally, we verify that

$$(X_1 X_2)(r) = X_1(r) X_2(r) = F_1 F_2 r^2 + G_1 G_2 (r + W(r)^2 - r) + (F_1 G_2 + F_2 G_1) r W(r),$$

which completes the proof.

Exercise 3.

Let $(W(t), t \geqslant 0)$ be a one-dimensional Brownian motion. Using the Itô formula:

i) compute $\mathrm{d}(W^m)$ where m is a positive integer,

ii) show that $Y(t) = e^{\lambda W(t) - \frac{\lambda^2 t}{2}}$ where $\lambda \in \mathbb{R}$ satisfies

$$\begin{cases} \mathrm{d}Y = \lambda Y \mathrm{d}W, \\ Y(0) = 1. \end{cases}$$

Solution

i) Using the Itô formula with $u(x) = x^m$, we have

$$\mathrm{d}(W^m) = u_x(W)\mathrm{d}W + \frac{1}{2} u_{xx}(W)\mathrm{d}t = m W^{m-1} \mathrm{d}W + \frac{1}{2} m(m-1) W^{m-2} \mathrm{d}t.$$

ii) It is clear that $Y(0) = 1$. We then apply the Itô formula with $u(t, x) = e^{\lambda x - \frac{\lambda^2 t}{2}}$ and get

$$\begin{aligned} \mathrm{d}u(t, W(t)) &= u_t(t, W(t))\mathrm{d}t + u_x(t, W(t))\mathrm{d}W + \frac{1}{2}u_{xx}(t, W(t))\mathrm{d}t \\ &= -\frac{\lambda^2}{2}u(t, W(t))\mathrm{d}t + \lambda u(t, W(t))\mathrm{d}W + \frac{1}{2}\lambda^2 u(t, W(t))\mathrm{d}t \\ &= \lambda u(t, W(t))\mathrm{d}W, \end{aligned}$$

which shows that $Y(t) = u(t, W(t))$ solves the equation.

Exercise 4.

Let $(X(t), t \geqslant 0)$ be an n-dimensional process with stochastic differential

$$\mathrm{d}X(t) = F(t)\mathrm{d}t + G(t)\mathrm{d}W(t),$$

where $F(t) \in \mathbb{R}^n$, $G(t) \in \mathbb{R}^{n \times m}$ and $W(t)$ is an m-dimensional Brownian motion. Consider $u(x) \colon \mathbb{R}^n \to \mathbb{R}$ defined as $u(x) = |x|^2 = \sum_{i=1}^n x_i^2$. Using the Itô formula, compute the stochastic differential $\mathrm{d}u(X(t))$ and give its expression in integral form.

Solution

As $u_{x_i}(x) = 2x_i$ and $u_{x_i x_j}(x) = 2\delta_{ij}$, the Itô formula gives

$$\mathrm{d}u(X(t)) = \sum_{i=1}^n 2X_i(t)\mathrm{d}X_i + \sum_{i=1}^n (GG^T)_{ii}\mathrm{d}t = 2X(t)^T\mathrm{d}X + \|G\|_F^2\mathrm{d}t,$$

where $\|A\|_F = \sqrt{\mathrm{trace}(A^T A)}$ is the Frobenius norm. Then, in integral form we have

$$u(X(s)) = u(X(r)) + \int_r^s (2X(t)^T F(t) + \|G\|_F^2)\mathrm{d}t + \int_r^s 2X(t)^T G(t)\mathrm{d}W(t).$$

Exercise 5.

Let $(W_1(t), t \geqslant 0)$ and $(W_2(t), t \geqslant 0)$ be two independent one-dimensional Brownian motions. Without using the Itô formula, show that

$$\mathrm{d}(W_1 W_2) = W_1 \mathrm{d}W_2 + W_2 \mathrm{d}W_1.$$

Hint. Consider $X(t) = \frac{1}{\sqrt{2}}(W_1(t) + W_2(t))$.

Solution

We verify that $X(t)$ is a Brownian motion as it is the sum of two $N(0, t/2)$ independent random variables, hence $X(t) \sim N(0, t)$. Moreover, using the properties of W_1 and W_2 we check that $\mathbb{E}(X(t)X(s)) = \min\{t, s\}$. Note that

$$X(t)^2 = \frac{W_1(t)^2}{2} + \frac{W_2(t)^2}{2} + W_1(t)W_2(t),$$

and that for any Brownian motion $W(t)$, we have seen that $\mathrm{d}(W^2) = 2W\mathrm{d}W + \mathrm{d}t$. Consequently,

$$\begin{aligned} \mathrm{d}(W_1 W_2) &= \mathrm{d}(X^2) - \frac{\mathrm{d}(W_1^2)}{2} - \frac{\mathrm{d}(W_2^2)}{2} \\ &= 2X\mathrm{d}X + \mathrm{d}t - W_1\mathrm{d}W_1 - \mathrm{d}t/2 - W_2\mathrm{d}W_2 - \mathrm{d}t/2 \\ &= (W_1 + W_2)(\mathrm{d}W_1 + \mathrm{d}W_2) - W_1\mathrm{d}W_1 - W_2\mathrm{d}W_2 \\ &= W_1\mathrm{d}W_2 + W_2\mathrm{d}W_1, \end{aligned}$$

which gives the result.

Exercise 6.

Let $B(t)$ be a one-dimensional standard Gaussian process such that $\mathbb{E}[B(t)B(s)] = \min\{s,t\}$. Show that $B(t)$ is a Brownian motion.
Recall. A real-valued stochastic process $(X(t), t \geqslant 0)$ is called a one-dimensional Gaussian process if for any integer $n \geqslant 1$ and any choice of times $t_1 < \ldots < t_n$ the random vector $(X(t_1), \ldots, X(t_n))$ has a multivariate Gaussian distribution. Moreover, it is called a standard one-dimensional Gaussian process if in addition $E[X(t)] = 0$ for all $t \geqslant 0$.

Solution

We have to show

(a) $B(0) = 0$ a.s.,

(b) $B(t) - B(s) \sim N(0, t-s)$ for all $t \geqslant s \geqslant 0$,

(c) $B(t_4) - B(t_3)$ and $B(t_2) - B(t_1)$ are independent for all $t_4 > t_3 \geqslant t_2 > t_1 \geqslant 0$.

For (a), note that by hypothesis $\mathbb{E}(B(0)^2) = 0$ and hence $B(0) = 0$ a.s. To show (b), we assume $s < t$ without loss of generality and note that

$$\begin{aligned}&\mathbb{E}(B(t) - B(s)) = 0,\\ &\mathbb{E}((B(t) - B(s))^2) = \mathbb{E}(B(t)^2) + \mathbb{E}(B(s)^2) - 2\mathbb{E}(B(t)B(s)) = t + s - 2s = t - s.\end{aligned}$$

Since for any Gaussian process the increments are Gaussian random variables (linear transformations of Gaussians are Gaussians), this ensures (b). For (c), we note that

$$\mathbb{E}((B(t_4) - B(t_3))(B(t_2) - B(t_1))) = t_2 - t_1 - t_2 + t_1 = 0.$$

For Gaussian random variables X, Y,

$$\mathbb{E}(XY) = \mathbb{E}(X)\mathbb{E}(Y) \ \Rightarrow \quad X, Y \text{ are independent}, \tag{6.1}$$

hence (c) is verified.