Exercises for Statistical analysis of network data – Sheet 6

1. Consider the network of the complete graph K_n on n vertices. Recall that the graph Laplacian is given by $L = \text{diag}(d_1, \ldots, d_n) - A = D - A$ and that the normalized graph Laplacian is $\mathcal{L} = D^{-1/2} L D^{-1/2}$.

   a) Calculate the degrees of this network by calculating $d = \mathbf{A}\mathbf{1}$.

   We note that the degrees are:

$$\begin{aligned} d &= (\mathbf{J}_n - \mathbf{I}_n)\,\mathbf{1} \\ &= (n-1)\mathbf{1}. \end{aligned}$$

   b) Calculate the graph Laplacian, and the normalized Laplacian for this network.

   We have that

$$\begin{aligned} L &= \text{diag}(d_1, \ldots, d_n) - (\mathbf{J}_n - \mathbf{I}_n) \\ &= (n-1)\mathbf{I}_n - (\mathbf{J}_n - \mathbf{I}_n) \\ &= n\mathbf{I}_n - \mathbf{J}_n. \end{aligned}$$

   In turn the normalized graph Laplacian takes the form of

$$\begin{aligned} \mathcal{L} &= D^{-1/2} L D^{-1/2} \\ &= \text{diag}((n-1)^{-1/2}, \ldots, (n-1)^{-1/2})\,\{n\mathbf{I}_n - \mathbf{J}_n\}\,\text{diag}((n-1)^{-1/2}, \ldots, (n-1)^{-1/2}) \\ &= \text{diag}((n-1)^{-1}, \ldots, (n-1)^{-1})\,\{n\mathbf{I}_n - \mathbf{J}_n\}. \end{aligned}$$

   c) For $n = 5$, multiply the normalized Laplacian by $\mathbf{e}_1 = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$.

   We find that

$$\begin{aligned} \mathcal{L}\mathbf{e}_1 &= \text{diag}((n-1)^{-1}, \ldots, (n-1)^{-1})\,\{n\mathbf{I}_n - \mathbf{J}_n\}\,\frac{1}{\sqrt{n}}\mathbf{1} \\ &= \frac{1}{\sqrt{n}(n-1)}\,\{n\mathbf{1} - n\mathbf{1}\} = 0. \end{aligned} \tag{1}$$

   This is an eigenvector with an eigenvalue of zero.

   d) The characteristic equation of a matrix $\{\mathcal{L}\}$ is given by

$$\|\mathcal{L} - \lambda\mathbf{I}\| = 0. \tag{2}$$

   Solve this equation in λ for the complete graph on 3 nodes. Factorize the characteristic equation down to the level you are able to.

   We wish to solve the equation

$$\begin{aligned} \|\text{diag}((n-1)^{-1}, \ldots, (n-1)^{-1})\,\{n\mathbf{I}_n - \mathbf{J}_n\} - \lambda\mathbf{I}\| &= 0 \\ \|\frac{1}{2}\left\{\begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix} - \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}\right\} - \lambda\mathbf{I}\| &= 0 \\ \|\begin{pmatrix} 1 & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & 1 & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & 1 \end{pmatrix} - \lambda\mathbf{I}\| &= 0 \\ \|\begin{pmatrix} 1-\lambda & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & 1-\lambda & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & 1-\lambda \end{pmatrix}\| &= 0. \end{aligned} \tag{3}$$

We therefore get

$$\begin{aligned}
(1-\lambda)\left((1-\lambda)^2-\frac{1}{4}\right)+\frac{1}{2}\left(-\frac{1}{2}(1-\lambda)+\frac{1}{4}\right)-\frac{1}{2}\left(\frac{1}{4}+\frac{1}{2}(1-\lambda)\right)&=0\\
(1-\lambda)^3-\frac{1}{4}(1-\lambda)-\frac{1}{4}(1-\lambda)+\frac{1}{8}-\frac{1}{8}-\frac{1}{4}(1-\lambda)&=0\\
(1-\lambda)^3-\frac{3}{4}(1-\lambda)&=0\\
(1-\lambda)\left(1-2\lambda+\lambda^2-\frac{3}{4}\right)&=0\\
(1-\lambda)\left(\frac{1}{4}-2\lambda+\lambda^2\right)&=0. \qquad (4)
\end{aligned}$$

The first root is $\lambda_0 = 1$. We have the final roots of $\lambda_0 = 1 \pm \sqrt{4-1} = 1 \pm \sqrt{3}$.

2. Consider a star on 4 nodes. The characteristic equation of a matrix $\mathcal{L}$ is given by

$$\|\mathcal{L}-\lambda\mathbf{I}\| = 0. \qquad (5)$$

Factorize the characteristic equation down to the level you are able to in λ.

Note that a star on four nodes has the adjacency matrix

$$\mathbf{A} = \begin{pmatrix} 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 1\\ 1 & 1 & 1 & 0 \end{pmatrix}$$

The degree vector is $d = \begin{pmatrix} 1 & 1 & 1 & 3 \end{pmatrix}^T$. We therefore find that the Laplacian is

$$\begin{aligned}
L &= \begin{pmatrix} 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 3 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 1\\ 1 & 1 & 1 & 0 \end{pmatrix}\\
&= \begin{pmatrix} 1 & 0 & 0 & -1\\ 0 & 1 & 0 & -1\\ 0 & 0 & 1 & -1\\ -1 & -1 & -1 & 3 \end{pmatrix}.
\end{aligned}$$

We can now calculate $\mathcal{L} = D^{-1/2}LD^{-1/2}$. We then have $D^{-1/2} = \begin{pmatrix} 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1/\sqrt{3} \end{pmatrix}$, and so we have

$$\begin{aligned}
\mathcal{L} = D^{-1/2}LD^{-1/2} &= \begin{pmatrix} 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1/\sqrt{3} \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & -1\\ 0 & 1 & 0 & -1\\ 0 & 0 & 1 & -1\\ -1 & -1 & -1 & 3 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1/\sqrt{3} \end{pmatrix}\\
&= \begin{pmatrix} 1 & 0 & 0 & -1\\ 0 & 1 & 0 & -1\\ 0 & 0 & 1 & -1\\ -1/\sqrt{3} & -1/\sqrt{3} & -1/\sqrt{3} & \sqrt{3} \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1/\sqrt{3} \end{pmatrix}\\
&= \begin{pmatrix} 1 & 0 & 0 & -1/\sqrt{3}\\ 0 & 1 & 0 & -1/\sqrt{3}\\ 0 & 0 & 1 & -1/\sqrt{3}\\ -1/\sqrt{3} & -1/\sqrt{3} & -1/\sqrt{3} & 1 \end{pmatrix}.
\end{aligned}$$

Then

$$
\begin{aligned}
&\|\mathcal{L}-\lambda\mathbf{I}\| = 0\\
\|&\begin{pmatrix} 1-\lambda & 0 & 0 & -1/\sqrt{3} \\ 0 & 1-\lambda & 0 & -1/\sqrt{3} \\ 0 & 0 & 1-\lambda & -1/\sqrt{3} \\ -1/\sqrt{3} & -1/\sqrt{3} & -1/\sqrt{3} & 1-\lambda \end{pmatrix}\| = 0.
\end{aligned} \tag{6}
$$

This equation becomes $(1-\lambda)\left|\begin{pmatrix} 1-\lambda & 0 & -1/\sqrt{3} \\ 0 & 1-\lambda & -1/\sqrt{3} \\ -1/\sqrt{3} & -1/\sqrt{3} & 1-\lambda \end{pmatrix}\right| + 1/\sqrt{3}\left|\begin{pmatrix} 0 & 1-\lambda & 0 \\ 0 & 0 & 1-\lambda \\ -1/\sqrt{3} & -1/\sqrt{3} & -1/\sqrt{3} \end{pmatrix}\right| = 0$.

3. Take as the adjacency matrix

$$
A = \begin{pmatrix} 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{pmatrix}.
$$

a) Calculate the degrees of this network by calculating $d = A\mathbf{1}$. We calculate

$$
\begin{aligned}
d &= A\mathbf{1} \qquad (7)\\
&= \begin{pmatrix} 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{pmatrix}\mathbf{1} \qquad (8)\\
&= \begin{pmatrix} 2 \\ 2 \\ 2 \\ 2 \\ 2 \\ 2 \end{pmatrix}. \qquad (9)
\end{aligned}
$$

b) Calculate the graph Laplacian, and the normalized Laplacian for this network.

$$
\begin{aligned}
L &= \mathrm{diag}(d_1, \dots, d_n) - \mathbf{A} \qquad (10)\\
&= 2\mathbf{I} - \begin{pmatrix} 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{pmatrix} \qquad (11)\\
&= \begin{pmatrix} 2 & -1 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 \\ -1 & -1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & -1 & -1 \\ 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & -1 & 2 \end{pmatrix}. \qquad (12)
\end{aligned}
$$

Thus

$$\mathcal{L} = D^{-1/2} L D^{-1/2} \tag{13}$$

$$= \frac{1}{2}\begin{pmatrix} 2 & -1 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 \\ -1 & -1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & -1 & -1 \\ 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & -1 & 2 \end{pmatrix}. \tag{14}$$

c) The characteristic equation of a matrix $\{\mathcal{L}\}$ is given by

$$\|\mathcal{L} - \lambda \mathbf{I}\| = 0. \tag{15}$$

Solve this equation in λ. Factorize the characteristic equation down to the level you are able to.

$$\begin{pmatrix} 1-\lambda & -1/2 & -1/2 & 0 & 0 & 0 \\ -1/2 & 1-\lambda & -1/2 & 0 & 0 & 0 \\ -1/2 & -1/2 & 1-\lambda & 0 & 0 & 0 \\ 0 & 0 & 0 & 1-\lambda & -1/2 & -1/2 \\ 0 & 0 & 0 & -1/2 & 1-\lambda & -1/2 \\ 0 & 0 & 0 & -1/2 & -1/2 & 1-\lambda \end{pmatrix} = 0. \tag{16}$$

We can therefore solve

$$\begin{pmatrix} 1-\lambda & -1/2 & -1/2 \\ -1/2 & 1-\lambda & -1/2 \\ -1/2 & -1/2 & 1-\lambda \end{pmatrix} = 0$$

$$(1-\lambda)((1-\lambda)^2 - 1/4) + (1/2)((-1/2)(1-\lambda) - 1/4) - (1/2)(1/4 + (1/2)(1-\lambda)) = 0.$$

d) For the values of λ determine the vectors of

$$\mathcal{L}\mathbf{e} = \lambda \mathbf{e}.$$

This is a straightforward calculation following on from the eigenvalues.

4. Take as the adjacency matrix

$$A = \begin{pmatrix} 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

a) Calculate the degrees of this network by calculating $d = A\mathbf{1}$. We note that

$$d = \begin{pmatrix} 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} \mathbf{1} = \begin{pmatrix} 4 \\ 4 \\ 4 \\ 3 \\ 4 \\ 1 \\ 2 \\ 2 \end{pmatrix}$$

b) Calculate the graph Laplacian, and the normalized Laplacian for this network. Follows the same steps as before.

c) The characteristic equation of a matrix $\{\mathcal{L}\}$ is given by

$$\|\mathcal{L} - \lambda\mathbf{I}\| = 0. \tag{17}$$

Solve this equation in λ. Factorize the characteristic equation down to the level you are able to. Follows the same steps as before.

d) For the values of λ determine the vectors of

$$\mathcal{L}\mathbf{e} = \lambda\mathbf{e}.$$

Follows the same steps as before.

e) Describe how to implement spectral clustering.

- Input: Adjacency matrix $A \in \mathbb{R}^{n\times n}$, number $k = 2$ of clusters to construct.
- Compute the unnormalized Laplacian L.
- Compute the first k eigenvectors $v_1, \dots, v_k$ of L
- Let $V \in \mathbb{R}^{n\times k}$ be the matrix containing vectors $v_1, \dots, v_k$ as columns.
- For $i = 1, \dots, n$ let y_i correspond to the ith row of V.
- Cluster the points $(y_i)_i$ in $\mathbb{R}^k$ with the k-means algorithm into clusters $\widetilde{C}_1, \dots, \widetilde{C}_k$.
- Outputs are clusters $C_1, \dots, C_k$ with

$$C_i = \{j | y_j \in \widetilde{C}_i\}.$$