Solutions for Statistical analysis of network data – Sheet 3

1. Is the graph C_4 the cycle graph on 4 nodes strictly balanced?

   Define $\bar{d} = 2e/n$. Define the maximum average degree of a graph F as

   $$m(F) = \max_{H \subset F}\{\bar{d}(H)\}.$$

   H is strictly balanced if $F \subset H$ and $d(F) = m(H)$ imply that $F = H$. The subgraphs of C_4 is P_2, P_3 and C_4 itself. $\bar{d} = 8/4 = 2$. On the other hand $\bar{d}$ for these subgraphs is $2/2 = 1, 4/3 = 1.333\ldots$ and C_4 is $8/4 = 2$. Equality implies the latter case, hence

2. Is the graph of a tadpole of a cycle with 5 nodes joined to a 3-node tail, e.g. a graph on 5+3-1 vertices strictly balanced?

   Define $\bar{d} = 2e/n = 14/7 = 2$. Subgraphs are $S_3, P_4, P_7 \ldots . \bar{d}$ is $6/4 = 3/2, 6/4$ and $12/7 < 2\ldots$ Thus it follows that it is not strictly balanced.

3. Count the number of triangles in the graph below, excluding the large connected components of the biggest connected component.

   The edges have nil triangles. The triangles have one each. K_4 has $4 \times 3 \times 2/6 = 4.K_5$ has $5 \times 4 \times 3/6 = 10$. The lollipop graph has the same as K_4 has $4 \times 3 \times 2/6 = 4$.

4. Assume that G is an Erdos-Renyi graph on n nodes. Determine the large sample approximation to the distribution of $X_{C_4}(G)$. Choose the success probability of the Erdos-Renyi graph so that a known limit arises.

   Suppose that H is a fixed strictly balanced graph with k vertices and $l \geq 2$ edges, and its automorphism group has a members. Let $c > 0$ be a constant and set $p = cn^{-k/l}$. For G generated as an ErdosRenyi graph with success probability p denote by $X_H(G)$ the number of copies of H in G. Then $X_H(G)$ asymptotically becomes a Poisson random variable with mean c^l/a. Here we recognise $l = 4$.

5. Assume that G is a Stochastic Blockmodel with 2 groups on n nodes. Determine the large sample approximation to the distribution of $X_{C_4}(G)$.

   We use the Theorem provided in the lecture. We denote the four probabilities $\theta_{11}, \theta_{12}, \theta_{21}, \theta_{22}$. Then take

   $$\lambda = \mathrm{E}X_H(G_n) = \binom{n}{4}\frac{4!}{8}\mu(H),$$

   with

   $$\mu(H) = \sum_{c_1,c_2,c_3,c_4}^{Q} h_{c_1}h_{c_2}h_{c_3}h_{c_4}\prod_{uv}\theta_{c_u c_v}.$$

   It has been shown by Coulson et al (2015) that the distribution of $X_H(G_n)$ becomes Poisson with mean λ.

6. What is the automorphosm group of K_4 ? How large it is?

   The automorphism group on K_n symmetric group S_n of degree n is the group of all permutations on n symbols.