Solutions for Statistical analysis of network data – Sheet 11

Here, 1 correspond to graph (a), 2 to (b) and 3 to (c). We list a few definitions that are helpful for any English longbowman that does not practice on Sundays:

- The *order* of a hypergraph is the number of nodes it has.
- In hypergraphs, if you have a hyperedge that connects a set of vertices, and another hyperedge that connects a subset of these vertices, the latter is said to be "*included*" in the former.
- A *multiple* edge occurs when there are two or more hyperedges connecting the same set of vertices.
- A hypergraph is *simple* if it has no multiple and induced edges.
- Nodes are *adjacent* if they are part of the same edge.
- The *degree* of a vertex is the cardinality of the set of edges it's belonging to.
- The *size* of an edge is its cardinality, i.e. its number of nodes.
- A hyper graph is *regular* is all its nodes have the same degree.
- A hyper graph is *uniform* is all its edges have the same cardinality.
- The *rank* of a hypergraph is the biggest edge cardinality in the hypergraph.

1. (a) Order $= 6, n_{\text{edges}} = 11$.

   (b) Included edges are edges that are a strict subset of another edge. Thus, edge $\{2,6\}$ in edge $\{2,4,6\}$ (top circle); edges $\{3,4\}$ and $\{5,6\}$ in edge $\{3,4,5,6\}$ (right-hand side circle); edge $\{1,5\}$ in edge $\{1,3,5\}$ (bottom circle); edges $\{3,4\}$ and $\{1,2\}$ in edge $\{1,2,3,4\}$ (lefthand side circle).

   (c) A multiple edge occurs when there are two or more hyperedges connecting the same set of vertices. Node groups that are encircled, that is, $\{2,4,6\}$ (top circle), $\{3,4,5,6\}$ (right-hand side circle), $\{1,3,5\}$ (bottom circle), and $\{1,2,3,4\}$ (left-hand side circle).

   (d) Graph (a) is not a simple graph: it has included edges as seen above as well as multiple edges.

   (e) Not adjacent: 1 and 6,2 and 5. All other pairs are adjacent.

   (f)
   - Vertex 1: $d_1 = 5$, neighbourhood: $\{2,3,4,5\}$
   - Vertex 2: $d_2 = 5$, neighbourhood: $\{1,3,4,6\}$
   - Vertex 3: $d_3 = 5$, neighbourhood: $\{1,2,4,5,6\}$
   - Vertex 4: $d_4 = 5$, neighbourhood: $\{1,2,3,5,6\}$
   - Vertex 5: $d_5 = 4$, neighbourhood: $\{1,3,4,6\}$
   - Vertex 6: $d_6 = 4$, neighbourhood: $\{2,3,4,5\}$

   (g) Edge list: $\{1,2\},\{1,3\},\{1,5\},\{2,4\},\{2,6\},\{3,4\},\{5,6\},\{2,4,6\},\{1,3,5\},\{3,4,5,6\}$, and $\{1,2,3,4\}$. The last two have cardinality 4 , the two before them have cardinality 3 , the others have cardinality 2 .

   (h) $\Delta(\mathcal{H}_a) = 5$.

   (i) Graph (a) is not regular (not all degrees are equal).

   (j) Graph (a) is not uniform (not all edges have the same cardinality).

   (k) The rank of graph correspond to the biggest cardinality of an edge in the hypergraph. Here, in (a), it is $r(\mathcal{H}_a) = 4$.

   (l) Isolated nodes cannot be found in either of the graphs (a), (b) and (c). Graph (c) contains a pendant node, the other two does not.

   (m) No singleton or empty edges can be found in any of the three graphs.

2. (a) Order $= 5, n_{\text{edges}} = 8$.

(b) Edge $\{1,2\}$ in edge $\{1,2,3\}$ and $\{1,2,5\}$;$\{1,3\}$ and $\{2,3\}$ are in edge $\{1,2,3\}$, $\{3,4\}$ and $\{4,5\}$ are in $\{3,4,5\}$

(c) Node groups that are encircled.

(d) Graph (a) is not a simple graph: it has included edges as seen above as well as multiple edges.

(e) Not adjacent: 1 and 4,2 and 4. All other pairs are adjacent.

(f)
- Vertex 1: $d_1 = 4$, neighbourhood: $\{2,3,5\}$
- Vertex 2: $d_2 = 4$, neighbourhood: $\{1,3,5\}$
- Vertex 3: $d_3 = 5$, neighbourhood: $\{1,2,4,5\}$
- Vertex 4: $d_4 = 3$, neighbourhood: $\{3,5\}$
- Vertex 5: $d_5 = 3$, neighbourhood: $\{1,2,3,4\}$

(g) Take the cardinality (i.e. number of nodes of question (b).

(h) $\Delta(\mathcal{H}_b) = 5$.

(i) Graph (b) is not regular (not all degrees are equal).

(j) Graph (b) is not uniform (not all edges have the same cardinality).

(k) The rank of graph (b) is $r(\mathcal{H}_a) = 3$.

(l) Isolated nodes cannot be found in either of the graphs (a), (b) and (c). Graph (c) contains a pendant node, the other two does not.

(m) No singleton or empty edges can be found in any of the three graphs.

3. (a) Order $= 4, n_{\text{edges}} = 4$.

(b) Edges $\{2,3\}$ and $\{3,4\}$ in edge $\{2,3,4\}$.

(c) Node groups that are encircled.

(d) Graph (c) is not a simple graph: it has included edges as seen above as well as multiple edges.

(e) Not adjacent: 1 and 4. All other pairs are adjacent.

(f)
- Vertex 1: $d_1 = 1$, neighbourhood: $\{2\}$
- Vertex 2: $d_2 = 3$, neighbourhood: $\{1,3,4\}$
- Vertex 3: $d_3 = 3$, neighbourhood: $\{2,4\}$
- Vertex 4: $d_4 = 2$, neighbourhood: $\{2,3\}$

(g) Edge list: $\{1,2\},\{2,3\},\{3,4\},\{2,3,4\}$ and the cardinality of each.

(h) $\Delta(\mathcal{H}_c) = 3$.

(i) Graph (c) is not regular (not all degrees are equal).

(j) Graph (c) is not uniform (not all edges have the same cardinality).

(k) The rank of graph (c) is $r(\mathcal{H}_a) = 3$.

(l) Isolated nodes cannot be found in either of the graphs (a), (b) and (c). Graph (c) contains a pendant node, the other two does not.

(m) No singleton or empty edges can be found in any of the three graphs.

(n) See Figure 1. for the two graphs. The edge set of the first is $e_1 = v_1v_2, e_2 = v_2v_3, e_3 = v_3v_4, e_4 = v_2v_3v_4$. The edge set of the second is $f_1 = w_1w_2, f_2 = w_2w_3, f_3 = w_3w_4, f_4 = w_1w_2w_3$. An isomorphism between the two is $\Phi(v_1) = w_4, \Phi(v_2) = w_3, \Phi(v_3) = w_2, \Phi(v_4) = w_1$. Since $\Phi(e_1) = w_3w_4 = f_3, \Phi(e_2) = w_2w_3 = f_2, \Phi(e_3) = w_1w_2 = f_1, \Phi(e_4) = w_1w_2w_3 = f_4$, and this constitutes a bijection, the two hypergraphs are isomorph.

(o) The incidence matrix of the graph on the left-hand side of Figure1, using the same edge labeling as in point (xv), is

$$I = \begin{array}{c|cccc} & e_1 & e_2 & e_3 & e_4 \\ \hline v_1 & 1 & 0 & 0 & 0 \\ v_2 & 1 & 1 & 0 & 1 \\ v_3 & 0 & 1 & 1 & 1 \\ v_4 & 0 & 0 & 1 & 1 \end{array}$$

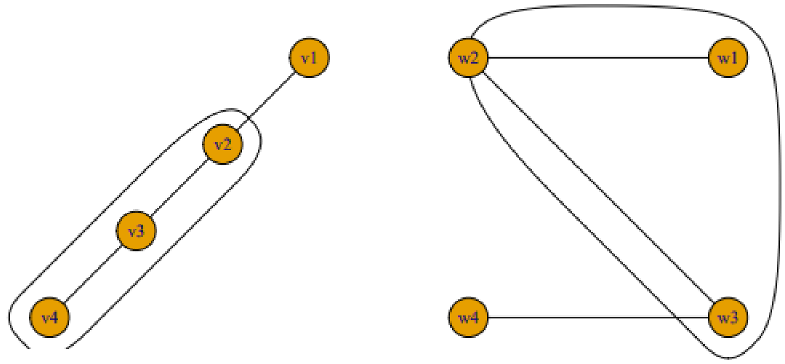

Figure 1: Isomorphism to different labelling.

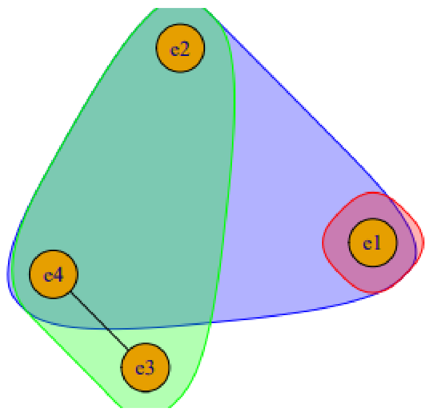

Figure 2: Dual of graph (c)

Its dual is the graph whose incidence matrix is the transpose of the above, that is,

$$I = \begin{array}{c|cccc} & v_1 & v_2 & v_3 & v_4 \\ \hline e_1 & 1 & 1 & 0 & 0 \\ e_2 & 0 & 1 & 1 & 0 \\ e_3 & 0 & 0 & 1 & 1 \\ e_4 & 0 & 1 & 1 & 1 \end{array}$$

The resulting graph is shown in Figure 2. The former edges now become vertices (see labels on vertices). There is an edge now with a single node (corresponding to the former node v_1, which was a pendant.