Exercises for Statistical analysis of network data – Sheet 10

1. For each pair of nodes, $\Pr\{ij \text{ exists}\} = \Pr\{ji \text{ exists}\} = \rho$. The situation is similar to asking for the distribution of the adjacency matrix of the projection of a multilevel network. The distribution is Bernoulli with the probability

$$\begin{aligned}\Pr\{ij \text{ exists or } ji \text{ exists}\} &= \Pr\{ij \text{ exists and } ji \text{ does not exist}\}\\ &\quad + \Pr\{ij \text{ does not exist and } ji \text{ exists}\} + \Pr\{ij \text{ exists and } ji \text{ exists}\}\\ &= 2\rho(1-\rho) + \rho^2.\end{aligned}$$

2. The "in"-graphon: $f^{(\mathrm{in})}(x) = 2x$, the "out"-graphon: $f^{(\mathrm{out})}(y) = 1$. There is a freedom to select the two functions, in principle $f^{(\mathrm{in})\prime}(x) = 2\lambda x$ and $f^{(\mathrm{out})\prime}(y) = \lambda^{-1}$ would do just as well.

   Assume that the densities of in-edges and out-edges are $\rho^{(\mathrm{in})}$ and $\rho^{(\mathrm{out})}$, respectively. Then, using the notations and definitions of slide 17 of the lecture, and denoting the latent variable corresponding to node i by x_i, we can write $\pi_i^{(\mathrm{in})} = 2\rho^{(\mathrm{in})}x_i$ and $\pi_i^{(\mathrm{out})} = \rho^{(\mathrm{out})}$. Therefore, $A_{ij} \sim \mathrm{Ber}(2x_i\rho^{(\mathrm{in})}\rho^{(\mathrm{out})})$. By the definitions of the in-degree and the out-degree as $d_i^{(\mathrm{in})} = \sum_{j\neq i} A_{ij}$ and $d_i^{(\mathrm{out})} = \sum_{j\neq i} A_{ji}$, it follows that

$$\begin{aligned}E\left\{d_i^{(\mathrm{in})}\right\} &= \sum_{j\neq i} E\left\{A_{ij}\right\} = 2(n-1)x_i\rho^{(\mathrm{in})}\rho^{(\mathrm{out})},\\ E\left\{d_i^{(\mathrm{out})}\right\} &= \sum_{j\neq i} E\left\{A_{ji}\right\} = \sum_{j\neq i} 2E\{x_i\}\rho^{(\mathrm{in})}\rho^{(\mathrm{out})} = (n-1)\rho^{(\mathrm{in})}\rho^{(\mathrm{out})},\\ \mathrm{Var}\left\{d_i^{(\mathrm{in})}\right\} &= \sum_{j\neq i} \mathrm{Var}\left\{A_{ij}\right\} = (n-1)2x_i\rho^{(\mathrm{in})}\rho^{(\mathrm{out})}\left(1-2x_i\rho^{(\mathrm{in})}\rho^{(\mathrm{out})}\right),\\ \mathrm{Var}\left\{d_i^{(\mathrm{out})}\right\} &= \sum_{j\neq i} \mathrm{Var}\left\{A_{ji}\right\} = (n-1)\rho^{(\mathrm{in})}\rho^{(\mathrm{out})}\left(1-\rho^{(\mathrm{in})}\rho^{(\mathrm{out})}\right).\end{aligned}$$