

Solutions for Statistical analysis of network data – Sheet 1

1. This corresponds to just noting the edges:

(a)

$$A_1 = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

This has as edge list $E_1 = \{(1, 2), (1, 3), (2, 3), (3, 4)\}$, and is known as a tadpole graph.

(b)

$$A_2 = \begin{pmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}.$$

This has as edge list $E_2 = \{(1, 2), (1, 3), (2, 4), (3, 4), (3, 5)\}$, and is known as a tadpole graph.

(c)

$$A_3 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

This has edge list $E_3 = \{(1, 2), (2, 3), (3, 4)\}$, and is known as a path. Paths are special cases of trees, and have fewer edges than nodes.

(d)

$$A_4 = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{pmatrix}.$$

This has edge list $E_4 = \{(1, 5), (2, 5), (3, 5), (4, 5)\}$. This is a star, and is also a special case of a tree.

(e)

$$A_5 = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}.$$

This has edge list $E_5 = \{(1, 2), (1, 3), (2, 4), (3, 4)\}$. This is a 4-cycle.

(f)

$$A_6 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}.$$

This is a special case of a tree as the number of edges is less than the number of nodes. This has edge list $E_6 = \{(1, 2), (2, 3), (3, 4), (3, 5)\}$.

2. The plots are at the end of the document. Adjacency matrices are given by

(a) $E_1 = \{(1, 2), (1, 3), (2, 3)\}$. This has adjacency matrix

$$A_1 = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

(b) $E_2 = \{(1, 5), (2, 5), (3, 5), (4, 5)\}$. This has adjacency matrix

$$A_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{pmatrix}.$$

(c) $E_3 = \{(1, 2), (2, 3), (3, 4), (4, 5)\}$. This has adjacency matrix

$$A_3 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

(d) $E_4 = \{(1, 2), (2, 3), (3, 4), (4, 5), (5, 6), (6, 1)\}$. This has adjacency matrix

$$A_4 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

A good check on whether something has been tabulated correctly is to take $A - A^T$. As the adjacency matrix is symmetric, this will normally uncover errors.

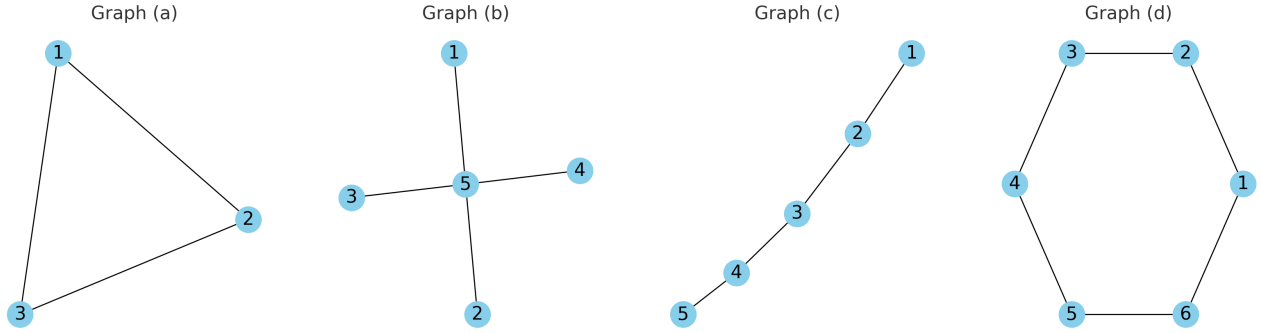


Figure 1: Plots of the graphs.

3. Assume that a_{ij} is generated from an Erdős-Rényi network with edge probability ρ .

- (a) Using moment generating functions we can note that the sum of $n - 1$ independent Bernoullis is $\text{Bin}(n - 1, \rho)$.
- (b) The expectation of a binomial is $(n - 1)\rho$. The variance is $(n - 1)\rho(1 - \rho)$.
- (c) The dispersion is $\text{Var}\{d_i\}/\text{E}\{d_i\} = (n - 1)\rho(1 - \rho)/((n - 1)\rho) = 1 - \rho$. We need $\rho = o(1)$.
- (d) For an inhomogeneous random graph we have what is known as a Poisson Binomial random variable, which is the distribution of $n - 1$ Bernoullis with different success probabilities. The expectation is $\sum_{i \neq j} p_{ij}$ and the variance is $\sum_{i \neq j} p_{ij}(1 - p_{ij})$.

4.

$$\text{ET} = \text{trace}(\text{E}(A^3))/6 = \frac{1}{6} \sum_{i=1}^n \sum_{j=1}^n \sum_{k=1}^n \text{E}(a_{ij}a_{jk}a_{ki}).$$

To determine the expectation we note that

$$E(a_{ij}a_{jk}a_{ki}) = E(a_{ij})E(a_{jk})E(a_{ki})I(i \neq j)I(i \neq k)I(k \neq j) \quad (1)$$

$$+ E(a_{ij})E(a_{jk})E(a_{ki})I(i = j)I(i \neq k)I(k \neq j) \quad (2)$$

$$+ E(a_{ij})E(a_{jk})E(a_{ki})I(i \neq j)I(i = k)I(k \neq j) \quad (3)$$

$$+ E(a_{ij})E(a_{jk})E(a_{ki})I(i \neq j)I(i \neq k)I(k = j) \quad (4)$$

$$= \rho^3 I(i \neq j)I(i \neq k)I(k \neq j). \quad (5)$$

Therefore

$$ET = \frac{1}{6} \sum_{i=1}^n \sum_{j=1}^n \sum_{k=1}^n \rho^3 I(i \neq j)I(i \neq k)I(k \neq j) \quad (6)$$

$$= \binom{n}{3} \cdot \rho^3. \quad (7)$$