
Series 13: convergence of moments

We recall the following result. Let X_n be a sequence of random variables. If for every $k \in \mathbb{N}$, $\mathbb{E}[X_n^k]$ converges to μ_k , and if moreover

$$(1) \quad \limsup_{k \rightarrow \infty} \frac{(\mu_{2k})^{1/2k}}{2k} < +\infty,$$

then X_n converges in distribution to the unique probability measure whose moments are the (μ_k) .

Exercise 1

Show that the moments of the standard Gaussian distribution $\mathcal{N}(0, 1)$ satisfy (1).

Exercise 2

Use the result recalled at the beginning of the series to show the following statements.

- 1) Let $X_1, X_2 \dots$ be i.i.d. random variables such that for every k , $E[|X_1|^k]$ is finite. Then $S_n = \sum_{i=1}^n X_i$ satisfies a central limit theorem.
- 2) For every $n \geq 1$, let $X_1^n, \dots, X_n^n$ be i.i.d. random variables such that

$$P[X_1^n = 1] = 1 - P[X_1^n = 0] = p_n,$$

where $p_n \in [0, 1]$. Show that if $np_n \rightarrow \lambda \in (0, +\infty)$ as n tends to infinity, then $\sum_{i=1}^n X_i^n$ converges in distribution to a Poisson random variable with parameter λ .

Exercise 3

Use characteristic functions to show the weak law of large numbers (i.e. convergence in probability under the assumption that the random variables are integrable).