Series 9: convergence in distribution **Solutions**

Exercise 1
Given $x \in \mathbb{R}$, we will write $x^+ = \max\{x, 0\}, x^- = (-x)^+$. Fix $n \in \mathbb{N}$; we have

$$1 = \int_{[f_n \le f_\infty]} f_\infty(x)dx + \int_{[f_n > f_\infty]} f_\infty(x)dx = \int_{[f_n \le f_\infty]} f_n(x)dx + \int_{[f_n > f_\infty]} f_n(x)dx,$$

so

$$\int_{[f_n \le f_\infty]} f_\infty(x)dx - \int_{[f_n \le f_\infty]} f_n(x)dx = \int_{[f_n > f_\infty]} f_n(x)dx - \int_{[f_n > f_\infty]} f_\infty(x)dx,$$

that is to say,

$$\int_{\mathbb{R}} (f_\infty - f_n)^+(x)dx = \int_{\mathbb{R}} (f_n - f_\infty)^+(x)dx,$$

since

$$x \in [f_n \le f_\infty]^c \implies (f_\infty - f_n)(x)^+ = 0,$$
$$x \in [f_n > f_\infty]^c \implies (f_n - f_\infty)(x)^+ = 0.$$

Let us write $F_n, 1 \leqslant n \leqslant \infty$ for the respective c.d.f. of $X_n, 1 \leqslant n \leqslant \infty$. Notice that, for every $x \in \mathbb{R}$,

$$\begin{aligned} |F_\infty(x) - F_n(x)| &= \left| \int_{-\infty}^x f_\infty(y)dy - \int_{-\infty}^x f_n(y)dy \right| \le \int_{\mathbb{R}} |f_\infty(y) - f_n(y)|dy \\ &= \int_{\mathbb{R}} (f_\infty - f_n)^+(y)dy + \int_{\mathbb{R}} (f_n - f_\infty)^+(y)dy = 2\int_{\mathbb{R}} (f_\infty - f_n)^+(y)dy. \end{aligned}$$

We have $0 \le (f_\infty - f_n)^+(y) \le f_\infty(y)$ and, by hypothesis, $(f_\infty - f_n)^+(y) \to 0$ as $n \to \infty$. Hence, by the dominated convergence theorem we have $\int_{\mathbb{R}} (f_\infty - f_n)^+(y)dy \to 0$, so $|F_\infty(x) - F_n(x)| \to 0$, and since x is arbitrary, we are done.
An alternative way to show that

$$\int_{\mathbb{R}} |f_\infty(y) - f_n(y)|dy$$

tends to 0 as n tends to infinity is to use Scheffé's theorem, see exercise 6 of series 3 (actually, the reasoning above contains a proof of Scheffé's theorem).

Exercise 2
Fix $\epsilon > 0$. Choose M such that $F(-M) < \epsilon, 1 - F(M) < \epsilon$. Since F is uniformly continuous on the compact set $[-M, M]$, there exists $\delta > 0$ such that $-M \le x, y \le M, |x - y| < \delta \implies |F(x) - F(y)| < \epsilon$. Now take a sequence $x_0, x_1, \ldots, x_k$ such that $x_0 = -M, x_k = M$ and for all $i, 0 < x_{i+1} - x_i < \delta$. Finally, choose N such that $n \ge N \implies |F(x_i) - F_n(x_i)| < \epsilon$ for all i. Fix $n \ge N, x \in \mathbb{R}$. We separately consider the cases:

- $x < -M$. We then have $F_n(x) \le F_n(-M) \le F(-M) + \epsilon < 2\epsilon$ and $F(x) \le F(-M) < \epsilon$, so $F_n(x), F(x) \in [0, 2\epsilon]$ and thus $|F_n(x) - F(x)| < 2\epsilon$.
- $x > M$. Similarly, we have $F_n(x), F(x) \in [1 - 2\epsilon, 1]$ and thus $|F_n(x) - F(x)| < 2\epsilon$.
- $x \in [-M, M]$. Choose i such that $x \in [x_i, x_{i+1}]$. Notice that

$$F(x_i) - \epsilon < F_n(x_i) \le F_n(x) \le F_n(x_{i+1}) < F(x_{i+1}) + \epsilon < F(x_i) + 2\epsilon;$$

$$F(x_i) \le F(x) \le F(x_{i+1}) < F(x_i) + \epsilon,$$

so $F_n(x), F(x) \in [F(x_i) - \epsilon, F(x_i) + 2\epsilon]$, thus $|F_n(x) - F(x)| \le 3\epsilon$.

This shows that $\sup_x |F_n(x) - F(x)| \le 3\epsilon$ when $n \ge N$; since ϵ is arbitrary, we are done.

Exercise 3

We will use the fact that if X is a random variable taking values in $\mathbb{Z}$, then F_X is constant, thus continuous, in $[a, a+1)$ for each $a \in \mathbb{Z}$.
Assume $X_n \to X$ in distribution. Fix $m \in \mathbb{Z}$, and notice that

$$\mathbb{P}(X_n = m) = F_{X_n}(m + 1/2) - F_{X_n}(m - 1/2) \overset{n\to\infty}{\longrightarrow} F_{X_\infty}(m + 1/2) - F_{X_\infty}(m - 1/2) = \mathbb{P}(X_\infty = m),$$

since $m + 1/2, m - 1/2$ are continuity points of F_{X_∞}.
For the converse, we have seen in the lecture that in the definition of the weak convergence of probability measures, it is equivalent to ask that it holds for every bounded bontinuous "test function", or for every continuous function with compact support. In other words, X_n converges to X_∞ in law if and only if for every $\varphi \in C_c(\mathbb{R})$ (the set of continuous functions with compact support), we have $\mathbb{E}[\varphi(X_n)] \to \mathbb{E}[\varphi(X_\infty)]$.
Let us fix $\varphi \in C_c(\mathbb{R})$. Since it has compact support, there exists $M > 0$ such that $\varphi = 0$ outside of $[-M, M]$. We thus have

$$\mathbb{E}[\varphi(X_n)] = \sum_{m=-M}^{M} \varphi(m)\, \mathbb{P}[X_n = m] \xrightarrow[n\to\infty]{} \sum_{m=-M}^{M} \varphi(m)\, \mathbb{P}[X_\infty = m] = \mathbb{E}[\varphi(X_\infty)],$$

and this finishes the proof.
Assume now that we only know the existence of

$$p_m := \lim_{n\to+\infty} \mathbb{P}[X_n = m].$$

If $\sum_{m\in\mathbb{Z}} p_m = 1$, then there exists a random variable X_∞ such that $p_m = \mathbb{P}[X_\infty = m]$, and we are back to the case considered above. But this need not be so, as can be seen by taking $X_n = n$. In such a case, X_n does not converge in probability.

Exercise 4

The first part of the statement was seen in the lecture. For the converse, assume $X_n \to X$ in distribution, where $X = c$ almost surely. Notice that $F_X(x) = 0$ for $x < c$ and $F_X(x) = 1$ for $x \ge c$, so every $x \ne c$ is a continuity point of F_x and then, by the hypothesis, we have $F_{X_n}(x) \to F_X(x)$ for every $x \ne c$. Fix $\epsilon > 0$. We have

$$\mathbb{P}(|X_n - c| > \epsilon) \le 1 - (F_{X_n}(c + \epsilon) - F_{X_n}(c - \epsilon)) \to 0,$$

so $X_n \to c$ in probability.

Exercise 5

We have seen that in order to check that $X_n + Y_n \xrightarrow[n\to\infty]{\text{law}} X + c$, it is enough to verify that for every $\varphi \in C_c(\mathbb{R})$ (the space of continuous functions with compact support), we have

$$\mathbb{E}[\varphi(X_n + Y_n)] \xrightarrow[n\to\infty]{} \mathbb{E}[\varphi(X + c)].$$

Let $\varphi \in C_c(\mathbb{R})$ and $\varepsilon > 0$. Since $\varphi \in C_c(\mathbb{R})$, it is bounded, and it is also uniformly continuous: there exists $\delta > 0$ such that

$$|y - x| \leqslant \delta \;\Rightarrow\; |\varphi(y) - \varphi(x)| \leqslant \varepsilon.$$

We now write

$$\mathbb{E}[\varphi(X_n + Y_n)] = \mathbb{E}[\varphi(X_n + Y_n)\mathbb{1}_{|Y_n - c| \leqslant \delta}] + \mathbb{E}[\varphi(X_n + Y_n)\mathbb{1}_{|Y_n - c| > \delta}].$$

Since φ is bounded and $Y_n \to c$ in probability (see exercise 4), the second term tends to 0 as n tends to infinity. For the first term, using uniform continuity, we obtain that

$$\left|\mathbb{E}[\varphi(X_n + Y_n)\mathbb{1}_{|Y_n - c| \leqslant \delta}] - \mathbb{E}[\varphi(X_n + c)\mathbb{1}_{|Y_n - c| \leqslant \delta}]\right| \leqslant \varepsilon.$$

Finally,

$$\limsup_{n\to\infty} \left|\mathbb{E}[\varphi(X_n + c)\mathbb{1}_{|Y_n - c|\leqslant\delta}] - \mathbb{E}[\varphi(X_n + c)]\right| \leqslant \limsup_{n\to\infty} \|\varphi\|_\infty \mathbb{P}[|Y_n - c| > \delta] = 0$$

since φ is bounded and $Y_n \to c$ in probability. We have thus shown that

$$\limsup_{n\to\infty} |\mathbb{E}[\varphi(X_n + Y_n)] - \mathbb{E}[\varphi(X_n + c)]| \leqslant \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, the limsup is in fact equal to 0. We conclude using the fact that

$$\mathbb{E}[\varphi(X_n + c)] \xrightarrow[n\to\infty]{} \mathbb{E}[\varphi(X + c)].$$

Exercise 6

We consider first the case when X_1 is uniformly distributed on $[0, 1]$. It is easy to see that $M_n \to 1$ a.s. as n tends to infinity. We can say more, and show that $n(1 - M_n)$ converges in distribution: for every $x \geqslant 0$,

$$\mathbb{P}[n(1 - M_n) \geqslant x] = \mathbb{P}[M_n \leqslant 1 - x/n] = (1 - x/n)^n \xrightarrow[n\to\infty]{} e^{-x}.$$

We have thus shown that $n(1 - M_n)$ converges in distribution to an exponential random variable of parameter 1. When X_1 follows a Cauchy distribution (whose density is $\pi^{-1}(1 + x^2)^{-1}\,\mathrm{d}x$), it is clear that M_n tends to infinity a.s. In fact, M_n/n converges in distribution:

$$\begin{aligned}
\mathbb{P}[M_n/n \leqslant x] &= \left(\frac{1}{\pi}\int_{-\infty}^{nx} \frac{1}{1 + x^2}\,\mathrm{d}x\right)^n \\
&= \left(\frac{1}{\pi}\left(\arctan(nx) + \frac{\pi}{2}\right)\right)^n \\
&= \left(\frac{1}{\pi}\left(\pi - \arctan\left(\frac{1}{nx}\right)\right)\right)^n \xrightarrow[n\to\infty]{} e^{-1/(\pi x)}.
\end{aligned}$$

The case of exponential distributions is treated similarly.

Exercise 7

• ρ is a metric.

The facts that $\rho(F, G) = \rho(G, F)$ and $\rho(F, G) \geq 0$ are quite immediate to verify. Noticing that, for any $\epsilon > 0, x \in \mathbb{R}$,

$$F(x - \epsilon) - \epsilon \leq F(x - \epsilon) \leq F(x) \leq F(x + \epsilon) \leq F(x + \epsilon) + \epsilon,$$

we see that $\rho(F, F) = 0$. Conversely, assume $\rho(F, G) = 0$. Then, there exists a sequence $\epsilon_n \searrow 0$ such that, for every x and n,

$$F(x - \epsilon_n) - \epsilon_n \leq G(x) \leq F(x + \epsilon_n) + \epsilon_n;$$

$$G(x - \epsilon_n) - \epsilon_n \leq F(x) \leq G(x + \epsilon_n) + \epsilon_n;$$

in particular, for every x and n we have

$$F(x) \leq G(x + \epsilon_n) + \epsilon_n, \quad G(x) \leq F(x + \epsilon_n) + \epsilon_n.$$

Letting $n \to \infty$, we get $F(x) \leq G(x+) = G(x)$ and $G(x) \leq F(x+) = F(x)$, so $F(x) = G(x)$. We have thus proved that $\rho(F, G) = 0$ if and only if $F = G$. Finally, for the triangle inequality, consider three distribution functions F, G, H and take sequences a_n, b_n such that $a_n \searrow \rho(F, G), b_n \searrow \rho(G, H)$ and, for each n,

$$G(x - a_n) - a_n \leq F(x) \leq G(x + a_n) + a_n;$$

$$H(x - b_n) - b_n \leq G(x) \leq H(x + b_n) + b_n.$$

We then have

$$F(x) \le G(x + a_n) + a_n \le H(x + a_n + b_n),$$

$$F(x) \ge G(x - a_n) - a_n) \ge H(x - a_n - b_n) - a_n - b_n,$$

so that $\rho(F, H) \le a_n + b_n$, then make $n \to \infty$ to get $\rho(F, H) \le \rho(F, G) + \rho(G, H)$.

• $\rho(F_n, F) \to 0$ implies $F_n \to F$ in distribution.
Fix $c_n \searrow 0$ such that, for every x and n,

$$F(x - c_n) - c_n \le F_n(x) \le F(x + c_n) + c_n.$$

Fixing x and letting $n \to \infty$, we get

$$F(x-) \le \liminf F_n(x) \le \limsup F_n(x) \le F(x+) = F(x),$$

so for all continuity points x of F, we have $\lim F_n(x) \to F(x)$, so $F_n \to F$ in distribution.

• $F_n \to F$ in distribution implies $\rho(F_n, F) \to 0$.
Fix $\epsilon > 0$. Find $A > 0$ such that $-A, A$ are continuity points of F and $F(-A), 1 - F(A) < \epsilon$. Now fix $x_0, x_1, \ldots, x_k$, all continuity points of F, such that $x_0 = -A, x_k = A, 0 < x_{i+1} - x_i < \epsilon$ for all i. Choose N such that $n \ge N$ implies $|F_n(x_i) - F(x_i)| < \epsilon$ for each i. Now fix $n \ge N$ and take $x \in \mathbb{R}$; we consider three cases:

- $x \in [-A, A]$. We can then take i such that $x \in [a_i, a_{i+1}]$. We have

  $$F_n(x) \le F_n(a_{i+1}) \le F(a_{i+1}) + \epsilon \le F(x + \epsilon) + \epsilon,$$

  and similarly,

  $$F_n(x) \ge F_n(a_i) \ge F(a_i) - \epsilon \ge F(x - \epsilon) - \epsilon.$$

- $x < -A$. Notice that, for every $y < -A$, we have $0 \le F(y), F_n(y) \le F(-A) + \epsilon \le 2\epsilon$. So we have

  $$F(x - 2\epsilon) - 2\epsilon \le 0 \le F_n(x) \le 2\epsilon \le F(x + 2\epsilon) + 2\epsilon.$$

- $x > A$. Similarly to the previous case, here we have

  $$F(x - 2\epsilon) - 2\epsilon \le F_n(x) \le F(x + 2\epsilon) + 2\epsilon.$$

The three cases together show that $\rho(F_n, F) < 2\epsilon$. Since ϵ is arbitrary, the proof is complete.