Series 8: almost sure convergence **Solutions**

Exercise 1

We use the strong law of large numbers, and adapt slightly the result on renewal theory seen in the lecture.

Exercise 2

Let $U_{n+1} = X_{n+1}/|X_n|$ (it is easy to check that almost surely, $X_n \neq 0$ for every n). The random variables (U_n) are independent and uniformly distributed on the unit disk with center 0. In particular, the distribution of U_n is the measure $\pi^{-1}\mathbb{1}_{x^2+y^2\leqslant 1}\,\mathrm{d}x\mathrm{d}y$. Moreover,

$$|X_n| = \prod_{k=1}^{n} |U_k|,$$

so $\ln(|X_n|)$ can be written as a sum of independend and identically distributed random variables, distributed as $\ln(|U_1|)$. The result then follows from the law of large numbers, provided $\ln(|U_1|)$ is integrable. We have:

$$E[\ln(|U_1|)] = \pi^{-1}\int_{\substack{0\leqslant r\leqslant 1\\ 0\leqslant\theta\leqslant 2\pi}} \ln(r)\, r\mathrm{d}r\mathrm{d}\theta = -\frac{1}{2}.$$

Hence, $\ln(|X_n|)/n$ converges almost surely to $-1/2$ (and in particular, $|X_n|$ converges almost surely to 0).

Exercise 3

Let R be the radius of convergence (a priori random) of the series. We will rely on the classical formula

$$R^{-1} = \limsup_{n\to+\infty} |X_n|^{1/n}.$$

Let $x \in \mathbb{R}$. Using Borel-Cantelli's lemma, one can show the following equivalence:

$$(1) \qquad R^{-1} \leqslant e^x \quad \text{a.s.} \quad \Leftrightarrow \quad \sum_{n=0}^{+\infty} P[|X_n|^{1/n} \geqslant e^x] < +\infty.$$

We have

$$\mathbb{P}[|X_n|^{1/n} \geqslant e^x] = \mathbb{P}[\log(|X_n|) \geqslant nx],$$

where we let $\log(0) = -\infty$. As soon as X_0 is not almost surely equal to 0, the sum

$$\sum_{n=0}^{+\infty} \mathbb{P}[\log(|X_n|) \geqslant nx]$$

diverges if $x < 0$. Hence, the radius of convergence of the series is at most 1. Let μ be the measure on $\mathbb{R}$ defined by

$$\mu([x, +\infty)) = \mathbb{P}[\log(|X_0|) \geqslant x]$$

(note that the measure μ is not a probability measure if $\mathbb{P}[X_0 = 0] > 0$.) Let us now assume that $x > 0$. We have

$$\sum_{n=0}^{+\infty} \mathbb{P}[\log(|X_n|) \geqslant nx] = \sum_{n=0}^{+\infty} \int_0^{+\infty} \mathbb{1}_{[nx,+\infty)}(u)\, \mathrm{d}\mu(u) \tag{2}$$
$$= \int_0^{+\infty} \sum_{n=0}^{+\infty} \mathbb{1}_{[nx,+\infty)}(u)\, \mathrm{d}\mu(u)$$
$$= \int_0^{+\infty} \lfloor u/x \rfloor\, \mathrm{d}\mu(u),$$

where $\lfloor u/x \rfloor$ is the integer part of u/x. We have the following alternative: either $\int_0^{+\infty} u\, \mathrm{d}\mu(u)$ is finite, in which case the sum in (2) is finite for every $x > 0$, and the radius of convergence is thus equal to 1 a.s. ; or the integral is infinite, the sum in (2) is also infinite for every $x > 0$, and the radius of convergence is equal to 0 a.s. We obtain the announced result since

$$\int_0^{+\infty} u\, \mathrm{d}\mu(u) = E[\log(|X_0|)\mathbb{1}_{\{|X_0| \geqslant 1\}}].$$

Exercise 4

Let

$$A = \{j : m \leqslant j < n \text{ and } |S_{m,j}| > 2a\},$$

and $\mathcal{A}$ be the event $A \neq \emptyset$. On this event, we define $J = \min A$, and let $J = -\infty$ on the complement of $\mathcal{A}$. We have

$$\mathcal{A} \text{ and } |S_{J,n}| \leqslant a \quad \Rightarrow \quad |S_{m,n}| = |S_{m,J} + S_{J,n}| > a,$$

so the probability of the event on the left-hand side is smaller than $P[|S_{m,n}| > a]$. This probability can be decomposed into

$$\sum_{j=m}^{n-1} P[J = j \text{ and } |S_{j,n}| \leqslant a]. \tag{3}$$

The events $J = j$ and $|S_{j,n}| \leqslant a$ are independent, so the sum in (3) is equal to

$$\sum_{j=m}^{n-1} P[J = j]\, P[|S_{j,n}| \leqslant a] \geqslant \sum_{j=m}^{n-1} P[J = j] \min_{m \leqslant k < n} P[|S_{k,n}| \leqslant a].$$

Finally, $\sum_{j=m}^{n-1} P[J = j] = P[A \neq \emptyset] = P[\max_{m \leqslant j < n} |S_{m,j}| > 2a]$, and we obtain the desired inequality.
For the second part, we will show that almost surely, $S_{0,n}$ is a Cauchy sequence. Let $\varepsilon, \delta > 0$. Using convergence in probability, one can check that there exists m such that for any $n, p \geqslant m$:

$$P[|S_{n,p}| > \varepsilon] \leqslant \delta. \tag{4}$$

Using the inequality proved in the first part, we obtain that for every $n \geqslant m$:

$$P[\max_{m \leqslant j < n} |S_{m,j}| > 2\varepsilon](1 - \delta) \leqslant P[\max_{m \leqslant j < n} |S_{m,j}| > 2\varepsilon] \min_{m \leqslant k < n} P[|S_{k,n}| \leqslant \varepsilon] \leqslant P[|S_{m,n}| > \varepsilon] \leqslant \delta.$$

We have shown that for every $n \geqslant m$:

$$P[\max_{m \leqslant j < n} |S_{m,j}| > 2\varepsilon] \leqslant \frac{\delta}{1 - \delta}.$$

As a consequence,

$$P[\exists n_0 : \max_{j \geqslant n_0} |S_{n_0,j}| \leqslant 2\varepsilon] \geqslant 1 - \frac{\delta}{1-\delta}$$

for every δ, so in fact the event on the left-hand side (let us write it C_ε), is of probability 1.
The event “the sequence $S_{0,n}$ is Cauchy” can be written

$$\bigcap_{\varepsilon>0} C_\varepsilon.$$

Clearly, one can restrict the above intersection to rational values of ε without changing the set. It is then a countable intersection of sets of measure 1, so it is itself of measure 1, and this finishes the proof.