Series 7: convergence in probability, strong law of large numbers **Solutions**

Exercise 1
Fix an arbitrary $\epsilon > 0$. Since $|X - Y| \leqslant |X - X_n| + |X_n - Y|$, then

$$\begin{aligned}\Big\{\omega \in \Omega : |X(\omega) - Y(\omega)| > \epsilon\Big\} &\subseteq \Big\{\omega \in \Omega : |X_n(\omega) - X(\omega)| + |X_n(\omega) - Y(\omega)| > \epsilon\Big\}\\ &\subseteq \Big\{\omega \in \Omega : |X_n(\omega) - X(\omega)| > \epsilon/2\Big\}\\ &\qquad\qquad\qquad \bigcup \Big\{\omega \in \Omega : |X_n(\omega) - Y(\omega)| > \epsilon/2\Big\} .\end{aligned}$$

Therefore,

$$P[\,|X - Y| > \epsilon\,] \quad \leqslant \quad P[\,|X_n - X| > \epsilon/2\,] \;+\; P[\,|X_n - Y| > \epsilon/2\,] \;\rightarrow\; 0\,, \quad \text{as } n \rightarrow \infty\,.$$

We have proved that $P[\,|X - Y| > \epsilon\,] = 0$ for any $\epsilon > 0$. Finally,

$$\{X \neq Y\} = \bigcup_{n=1}^{+\infty} \{|X - Y| > 1/n\},$$

which is a sequence of increasing events. We can thus conclude that

$$P[X \neq Y] = \lim_{n\to\infty} P[|X - Y| > 1/n] = 0,$$

which means that $X = Y$ a.s..

Exercise 2
Recall the following fact from Real Analysis:

Lemma 1. *Let $(a_n)_{n\geqslant 1}$ be a sequence of real numbers and let $a \in \mathbb{R}$. $a_n \to a$ if and only if for every subsequence $(a_{n_k})_{k\geqslant 1}$ there is a further subsequence $(a_{n_{k_\ell}})_{\ell\geqslant 1}$ that converges to a.*

Fix an arbitrary $\epsilon > 0$. It suffices to check that $P[\,|X_n - Z| > \epsilon\,] \to 0$ as $n \to \infty$. For it, we apply the above lemma for the sequence of real numbers

$$(P[\,|X_n - Z| > \epsilon\,] : n \geqslant 1).$$

Let us fix an arbitrary subsequence $(P[\,|X_{n_k} - Z| > \epsilon\,] : k \geqslant 1)$. By hypothesis, there exists a further subsequence $(X_{n_{k_\ell}})_{\ell\geqslant 1}$ such that $X_{n_{k_\ell}} \xrightarrow{P} Z$ and then $P[\,|X_{n_{k_\ell}} - Z| > \epsilon\,] \to 0$ as $\ell \to \infty$. Finally, Lemma 1 applies and we get $P[\,|X_n - Z| > \epsilon\,] \to 0$.

Exercise 3
We shall apply Exercice 2 to the sequence $f(X_n)$. So, fix an arbitrary subsequence $(f(X_{n_k}))_{k\geqslant 1}$. Since $X_n \xrightarrow{\text{P}} X$, we have $X_{n_k} \xrightarrow{\text{P}} X$. Furthermore, there exists a subsequence $(X_{k_\ell})_{\ell\geqslant 1}$ such that $X_{n_{k_\ell}} \to X$ almost surely as $\ell \to \infty$. Since f is continuous, we have $f(X_{n_{k_\ell}}) \to f(X)$ almost surely. In particular, $f(X_{n_{k_\ell}}) \xrightarrow{\text{P}} f(X)$ and finally the desired result follows from Exercice 2.

Exercise 4
The assertion follows from the following observation:

$$\frac{X_n}{n} \quad = \quad \frac{S_n}{n} - \Big(\frac{n-1}{n}\Big)\frac{S_{n-1}}{n-1}\,, \quad \forall n \geqslant 2\,,$$

and from the fact that both terms in this difference converges to 0 in probability.

Exercise 5
For each $i \geq 1$, we have

$$\mathbb{E}(X_i^2) = \mathrm{Var}(X_i) = 1;$$
$$\mathbb{E}((X_i - 1)^2) = \mathbb{E}(X_i^2) + 1 - 2\mathbb{E}(X_i) = 2.$$

Since $X_1^2, X_2^2, \ldots$ are independent and integrable, the strong law of large numbers implies that the event

$$A = \left\{ \frac{X_1^2 + \ldots + X_n^2}{n} \to 1 \right\}$$

has probability 1. Similarly, $(X_1 - 1)^2, (X_2 - 1)^2, \ldots$ are independent and integrable, so the event

$$B = \left\{ \frac{(X_1 - 1)^2 \ldots + (X_n - 1)^2}{n} \to 2 \right\}$$

has probability 1. Then, $\mathbb{P}(A \cap B) = 1$ and $\omega \in A \cap B$ implies

$$\frac{X_1^2(\omega) + \ldots + X_n^2(\omega)}{(X_1(\omega) - 1)^2 + \ldots + (X_n(\omega) - 1)^2}$$
$$= \frac{X_1^2(\omega) + \ldots + X_n^2(\omega)}{n} \cdot \frac{n}{(X_1(\omega) - 1)^2 + \ldots + (X_n(\omega) - 1)^2} \xrightarrow[n\to\infty]{} \frac{1}{2}.$$

Exercise 6
X_1 is integrable:

$$E[X_1] = \int_{\mathbb{R}} x f(x)\, dx = \int_{-1/2}^{+\infty} x e^{-(x+1/2)}\, dx = \frac{1}{2}.$$

Then, by the law of large numbers, $\frac{S_n}{n} \to 1/2$ almost surely, and thus $S_n \to +\infty$ almost surely.

Exercise 7
A simple computation provides

$$E[\ \ln(X_1)\ ] = \int_0^1 \ln(u) du = -1\,.$$

Since

$$\ln\left(\left(\prod_{i=1}^n X_i \right)^{1/n} \right) = \frac{\sum_{i=1}^n \ln(X_i)}{n}$$

and $(\ \ln(X_i))_{i\geqslant 1}$ are i.i.d., by the strong law of large numbers, we have

$$\ln\left(\left(\prod_{i=1}^n X_i \right)^{1/n} \right) \to -1\,, \quad \text{a.s.}$$

which in turn implies that

$$\left(\prod_{i=1}^n X_i \right)^{1/n} \to e^{-1}\,, \quad \text{a.s.}$$

Exercise 8
We will take $\Omega = (0, 1]$ with Borel σ-algebra and Lebesgue measure.

For $n \geq 1$ and $i \in \{2^n, \ldots, 2^{n+1}-1\}$, define

$$X_i = I_{\left(\frac{i-2^n}{2^n}, \frac{i-2^n+1}{2^n}\right]},$$

where I denotes the indicator function. Notice that if $j \geq 2^n$, then X_j is the indicator function of an interval of length less than 2^{-n}, and this implies that X_i converges to 0 in probability. Now, notice that for each $x \in (0,1]$, there are infinitely many values of i such that $X_i(x) = 1$, so we can define by induction

$$N_0 \equiv 0; \qquad N_{n+1}(x) = \inf\{i > N_n(x) : X_i = 1\} \quad (x \in (0,1],\ n \geq 0).$$

Hence, by construction we have $X_{N_n}(x) = 1$ for all x and all n, and in particular $X_{N_n} \to 1$ almost surely as $n \to \infty$.

Exercise 9

1) Let $i_1, \ldots, i_{n-1}$ be a sequence of strictly positive integers. We can write the probability of the event

$$\forall k \leqslant n-1 : \tau^n_{k+1} - \tau^n_k = i_k$$

as the following sum

$$\sum_{\sigma \in S_n} P[X_1 = \sigma(1) \text{ et } \forall k \leqslant n-1 :$$

$$X_{i_1+\cdots+i_{k-1}+2}, \ldots, X_{i_1+\cdots+i_k} \in \{\sigma(1), \ldots, \sigma(k)\} \text{ et } X_{i_1+\ldots+i_k+1} = \sigma(k+1)],$$

where S_n is the symmetric group with n elements. By independence of the X_i, this probability is equal to

$$\sum_{\sigma \in S_n} \frac{1}{n} \prod_{k=1}^{n-1} P\left[X_{i_1+\cdots+i_{k-1}+2} \in \{\sigma(1), \ldots, \sigma(k)\}\right] \cdots$$

$$P\left[X_{i_1+\cdots+i_k} \in \{\sigma(1), \ldots, \sigma(k)\}\right] P\left[X_{i_1+\ldots+i_k+1} = \sigma(k+1)\right].$$

This expression is finally equal to

$$\sum_{\sigma \in S_n} \frac{1}{n} \prod_{k=1}^{n-1} \left(\frac{k}{n}\right)^{i_k - 1} \frac{1}{n} = \prod_{k=1}^{n-1} \left(\frac{k}{n}\right)^{i_k-1} \left(1 - \frac{k}{n}\right).$$

We recognize here the distribution of a sequence of independent geometric random variables, with respective parameters $1 - k/n$.

2) Using the above result, together with the fact that

$$T_n = \sum_{k=1}^{n-1} (\tau^n_{k+1} - \tau^n_k),$$

we can estimate the expectation and the variance of T_n:

$$E[T_n] = \sum_{k=1}^{n-1} \frac{1}{1-k/n} = n \sum_{k=1}^{n-1} \frac{1}{n-k} \sim n \log(n),$$

$$\operatorname{Var}(T_n) = n \sum_{k=1}^{n-1} \frac{k}{(n-k)^2} \leqslant n^2 \sum_{k=1}^{+\infty} \frac{1}{k^2}.$$

Let $\epsilon > 0$. Chebyshev's inequality tells us

$$P[|T_n - E[T_n]| > \epsilon n \log(n)] \leqslant \frac{\operatorname{Var}(T_n)}{\epsilon^2 n^2 \log^2(n)} \xrightarrow[n\to\infty]{} 0,$$

which proves that $|T_n - E[T_n]|/(n \log(n))$ tends to 0 in probability. The conclusion comes noting that $E[T_n]/(n \log(n))$ tends to 1 when n tends to infinity.

Remark. Erdős and Rényi (1961) have shown that $T_n/n - \log(n)$ converges to a simple limit distribution.