**Series 13: convergence of moments** **Solutions**

**Exercise 1**
Let $X$ follow the standard Gaussian law. We recall that

$$\mathbb{E}X^{2n} = \frac{(2n)!}{2^n n!} =: \mu_{2n}.$$

Using Stirling's formula, we obtain

$$\mu_{2k} \sim 2^{k+1/2} k^k e^{-k},$$

and thus $(\mu_{2k})^{1/2k} \sim \sqrt{2k/e} = O(2k)$.

**Exercise 2**
Possibly centering and renormalizing the $X_i$, we can assume that $E[X_1] = 0$ and $E[(X_1)^2] = 1$. We will write $\mu_k = E[(X_1)^k]$. We would like to compute the moments of $S_n = \sum_{i=1}^n X_i$.

$$E\left[\left(\sum_{i=1}^n X_i\right)^k\right] = \sum_{\underline{i} \in \{1,\dots,n\}^k} \underbrace{\mathbb{E}[X_{i_1} \cdots X_{i_k}]}_{:=a_{\underline{i}}},$$

where we write $\underline{i} = (i_1, \dots, i_k)$. Since the random variables are i.i.d., the quantity $a_{\underline{i}}$ only depends on the number of times each index is repeated. Let $j_1 < \cdots < j_l$ be the distinct elements appearing in $\underline{i}$, so that the sets $\{i_1, \dots, i_k\}$ and $\{j_1, \dots, j_l\}$ are equal. Then $\underline{j} = \{j_1, \dots, j_l\}$ belongs to $\mathcal{P}_n^l$, the collection of subsets of $\{1, \dots n\}$ with $l$ elements. Let $m_i$ be the number of times (at least equal to 1) that $j_i$ appears in the sequence $i_1, \dots, i_k$. Note that $\underline{m} = (m_1, \dots, m_l)$ takes values in the set

$$\mathcal{M}_l = \left\{(m'_1, \dots, m'_l) \in (\mathbb{N}^*)^l : \sum_{i=1}^l m'_i = k\right\}.$$

We have thus defined an application

$$\varphi : \left\{ \begin{array}{ccc} \{1, \dots n\}^k & \to & \bigcup_{l=1}^k \mathcal{M}_l \times \mathcal{P}_n^l \\ \underline{i} & \mapsto & (\underline{m}, \underline{j}). \end{array} \right.$$

Note that if $\varphi(\underline{i}) = (\underline{m}, \underline{j})$, then

$$a_{\underline{i}} = E[(X_{j_1})^{m_1} \cdots (X_{j_l})^{m_l}] = \mu_{m_1} \cdots \mu_{m_l},$$

and thus $a_{\underline{i}}$ depends in fact only on $\underline{m}$. By abuse of notation, we will write $a_{\underline{i}} = a_{\underline{m}}$. We thus obtain the following decomposition:

$$\sum_{\underline{i} \in \{1,\dots,n\}^k} a_{\underline{i}} = \sum_{l=1}^k \sum_{(\underline{m},\underline{j}) \in \mathcal{M}_l \times \mathcal{P}_n^l} |\varphi^{-1}(\underline{m}, \underline{j})| \, a_{\underline{m}}.$$

We now need to compute $|\varphi^{-1}(\underline{m}, \underline{j})|$. A combinatorial argument show that this number is the multinomial coefficient $C_{\underline{m}}$, which is given by

$$C_{\underline{m}} = \frac{k!}{m_1! \cdots m_l!},$$

but for our purpose, we only need to see that it does not depend on $\underline{j}$, and to know $C_{2,\dots,2}$. We thus obtain

$$\sum_{\underline{i}\in\{1,\dots,n\}^k} a_{\underline{i}} = \sum_{l=1}^{k} C_n^l \sum_{\underline{m}\in\mathcal{M}_l} C_{\underline{m}}\, a_{\underline{m}}, \tag{1}$$

where $C_n^l$ is the usual binomial coefficient.
We have now done the most difficult part of the work ! Note that if one of the $m_i$'s is 1, then $a_{\underline{m}} = 0$ since $\mu_1 = 0$. In other words, in order for $a_{\underline{m}}$ to be non-zero, it must be that all the $m_i$'s are at least equal to 2. This implies that in the sum over $l$ in (1), we can restrict ourselves to the $l$'s such that $2l \leqslant k$.
If $k$ is odd, say $k = 2q + 1$, then we obtain that $l \leqslant q$. Since moreover $C_n^l = O(n^l)$, it comes that

$$E\left[\left(\sum_{i=1}^{n} X_i\right)^{2q+1}\right] = O(n^q),$$

which implies that

$$E\left[\left(n^{-1/2}\sum_{i=1}^{n} X_i\right)^{2q+1}\right] \xrightarrow[n\to\infty]{} 0. \tag{2}$$

On the other hand, if $k$ is even, say $k = 2q$, then for the same reason, the term corresponding to $l = q$ is the only non-negligible one, and we obtain:

$$E\left[\left(\sum_{i=1}^{n} X_i\right)^{2q}\right] \sim C_n^q\, C_{\underbrace{(2,\dots,2)}_{q \text{ times}}} (\mu_2)^q,$$

from which it follows that

$$E\left[\left(n^{-1/2}\sum_{i=1}^{n} X_i\right)^{2q}\right] \xrightarrow[n\to\infty]{} \frac{(2q)!}{2^q\, q!}. \tag{3}$$

The moments obtained in (2) and (3) are indeed the moments of a standard Gaussian random variable, which is what we wanted to prove.

**Exercise 3**
Let $X_1, X_2, \dots$ be i.i.d. integrable random variables. We can assume that $E[X_1] = 0$. Since $X_1$ is in $L^1$, the random variable

$$\left|\frac{\mathrm{d}}{\mathrm{d}t} e^{itX_1}\right| = |X_1|$$

is uniformly dominated over $t$ by an integrable random variable, so the characteristic function of $X_1$ (let us write it $\varphi$) is differentiable on $\mathbb{R}$, and with derivative

$$\varphi'(t) = iE[X_1 e^{itX_1}].$$

In particular, $\varphi'(0) = 0$. In other words, $\eta$ defined by

$$\eta(t) = \frac{\varphi(t) - 1}{t}$$

tends to 0 as $t$ tends to 0.
The characteristic function of $\sum_{i=1}^n X_i/n$, evaluated at $t$, is equal to

$$\varphi(t/n)^n = \left(1 + \frac{t}{n}\eta\left(\frac{t}{n}\right)\right)^n.$$

The function $\eta$ taking its values in the complex plane, some care is necessary before using the logarithm, but since $\eta$ tends to 0 at 0, one can justify the following computation:

$$\varphi(t/n)^n = \exp\left(n \log\left(1 + \frac{t}{n}\eta\left(\frac{t}{n}\right)\right)\right) = \exp\left(n\frac{t}{n}\eta\left(\frac{t}{n}\right)(1 + o(1))\right) \xrightarrow[n\to\infty]{} 1,$$

and this proves the result.