Series 11: characteristic functions and central limit theorem **Solutions**

Exercise 1
($i.$)

$$\varphi_{X_n+a_n}(t) = e^{ita_n}\varphi_{X_n}(t) \to e^{ita}\varphi(t),$$

where φ is the characteristic function associated to $\mathcal{N}(0,1)$; the convergence follows from the hypothesis. Now notice that $e^{ita}\varphi(t)$ is the characteristic function associated to the law of $X + a$, where $X \sim \mathcal{N}(0,1)$, that is, associated to $\mathcal{N}(a,1)$. We thus have $X_n + a_n \xrightarrow{(\mathcal{L})} N(a,1)$.
($ii.$) Since φ_X is continuous and $\varphi_x(0) = 1$, we can choose $t \neq 0$ such that $\varphi_X(t) \neq 0$. Since X_n converges in distribution to some X, we have $\varphi_{X_n}(t) = e^{-\sigma_n^2 t^2/2} \to \varphi_X(t)$. This shows that $\varphi_X(t)$ is real and strictly positive. Thus, we have $\log(e^{-\sigma_n^2 t^2/2}) \to \log(\varphi_X(t))$, and this is only possible if σ_n converges to some finite non-negative value.

Exercise 2
The normal distribution has all finite moments. This implies that φ is infinitely differentiable and, for each $n \geq 0$,

$$\varphi^{(n)}(0) = i^n\mathbb{E}(X^n).$$

Doing infinite Taylor expansion around zero, this gives for every $t \in \mathbb{R}$

$$\varphi(t) = \sum_{n=0}^{\infty} \frac{i^n\mathbb{E}(X^n)}{n!}t^n.$$

By symmetry of the distribution, all moments of odd orders are zero, so the above is equal to $\sum_{n=0}^{\infty} \frac{i^{2n}\mathbb{E}(X^{2n})}{(2n)!}t^{2n} = \sum_{n=0}^{\infty} \frac{(-1)^n\mathbb{E}(X^{2n})}{(2n)!}t^{2n}$. On the other hand, we know that $\varphi(t) = e^{-t^2/2}$, and using the usual exponential series $e^a = \sum_{n=0}^{\infty} \frac{a^n}{n!}$, we have

$$\varphi(t) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^n(n!)}t^{2n}.$$

We now use unicity of Taylor expansion to get, for every n,

$$\frac{(-1)^n}{(2n)!}\mathbb{E}(X^{2n}) = \frac{(-1)^n}{2^n(n!)}t^{2n}.$$

and the result follows.

Exercise 3
Recall that if $X_1, X_2, \ldots$ are i.i.d. Poisson random variables of parameter 1, then $S_n = X_1 + \ldots + X_n$ has a Poisson law with parameter n (this can be proved for instance by an explicit computation or by using characteristic functions). Note also that $\mathbb{E}(S_n) = \mathrm{Var}(S_n) = n$. Hence,

$$e^{-n}\sum_{k=0}^{n} \frac{n^k}{k!} = \mathbb{P}(\mathrm{Poi}(n) \leq n) =$$

$$\mathbb{P}(S_n \leq n) = \mathbb{P}\left(\frac{S_n - n}{\sqrt{n}} \leq \frac{n-n}{\sqrt{n}}\right) \to \mathbb{P}(\mathcal{N}(0,1) \leq 0) = \frac{1}{2}.$$

Exercise 4
Let us define R_n such that
$$S_n = n + \sqrt{n} R_n.$$
By the central limit theorem, we know that R_n converges in law to $\mathcal{N}(0, \sigma^2)$ as n tends to infinity. Moreover,
$$\begin{aligned} \mathbb{P}[\sqrt{S_n} - \sqrt{n} \leqslant x] &= \mathbb{P}[S_n \leqslant (x + \sqrt{n})^2] \\ &= \mathbb{P}\left[R_n \leqslant \frac{(x+\sqrt{n})^2 - n}{\sqrt{n}}\right] \\ &= \mathbb{P}\left[R_n \leqslant \frac{x^2}{\sqrt{n}} + 2x\right]. \end{aligned}$$
Since R_n converges in law to $\mathcal{N}(0, \sigma^2)$, we know that $R_n - x^2/\sqrt{n}$ also converges in law to $\mathcal{N}(0, \sigma^2)$. Thus, letting $N \sim \mathcal{N}(0, \sigma^2)$, the probability above converges to
$$\mathbb{P}[N \leqslant 2x] = \mathbb{P}[N/2 \leqslant x].$$
We have thus proved that $\sqrt{S_n} - \sqrt{n}$ converges in law to $N/2$, which has distribution $\mathcal{N}(0, \sigma^2/4)$.

Exercise 5
Let
$$\sigma_n = \left(\sum_{k=1}^n X_k^2\right)^{-1/2}.$$
By the strong law of large numbers, we have
$$\sigma_n \xrightarrow[n\to\infty]{\text{a.s.}} \sigma := \mathbb{E}[X_1^2]^{1/2}.$$
Let $\delta > 0$, sufficiently small so that $\sigma - \delta > 0$, and let E_n be the event
$$\{|\sigma_n - \sigma| \leqslant \delta\}.$$
By the above observation, we know that $\mathbb{P}[E_n]$ tends to 1 as n tends to infinity. Let us write $S_n = X_1 + \ldots + X_n$. For every $x \in \mathbb{R}$, we compute
$$\mathbb{P}[S_n/\sigma_n \leqslant x] \leqslant \mathbb{P}[S_n \leqslant (\sigma + \delta)x, E_n] + \mathbb{P}[E_n^c],$$
where we write E_n^c for the complementary of E_n. Hence,
$$\limsup_{n\to\infty} \mathbb{P}[S_n/\sigma_n \leqslant x] \leqslant \limsup_{n\to\infty} \mathbb{P}[S_n \leqslant (\sigma+\delta)x, E_n].$$
But
$$\mathbb{P}[S_n \leqslant (\sigma+\delta)x, E_n] \leqslant \mathbb{P}[S_n \leqslant (\sigma+\delta)x] - \mathbb{P}[E_n^c],$$
so
$$\limsup_{n\to\infty} \mathbb{P}[S_n/\sigma_n \leqslant x] \leqslant \limsup_{n\to\infty} \mathbb{P}[S_n \leqslant (\sigma+\delta)x].$$
By the central limit theorem, the right-hand side is equal to $\mathbb{P}[N \leqslant (\sigma+\delta)x]$, where $N \sim \mathcal{N}(0, \sigma^2)$. We have thus shown
$$\limsup_{n\to\infty} \mathbb{P}[S_n/\sigma_n \leqslant x] \leqslant \mathbb{P}[N \leqslant (\sigma+\delta)x].$$
Since $\delta > 0$ can be taken arbitrarily small, this actually show that
$$\limsup_{n\to\infty} \mathbb{P}[S_n/\sigma_n \leqslant x] \leqslant \mathbb{P}[N \leqslant \sigma x] = \mathbb{P}[N/\sigma \leqslant x].$$

By the same reasoning, we obtain that

$$\liminf_{n\to\infty} \mathbb{P}[S_n/\sigma_n \leqslant x] \geqslant \mathbb{P}[N/\sigma \leqslant x],$$

and thus

$$\lim_{n\to\infty} \mathbb{P}[S_n/\sigma_n \leqslant x] = \mathbb{P}[N/\sigma \leqslant x].$$

Since $N/\sigma \sim \mathcal{N}(0,1)$, this proves the claim.

Exercise 6
(i) Let

$$Z = \limsup_{n\to\infty} \frac{S_n}{\sqrt{n}}.$$

The random variable Z is well-defined (with values in $\mathbb{R} \cup \{\pm\infty\}$) and measurable with respect to $\sigma(X_1, X_2, \ldots)$, with $X_1, X_2, \ldots$ independent random variables. Let k be a positive integer. We have

$$Z = \limsup_{n\to\infty} \frac{S_k + (S_n - S_k)}{\sqrt{n}} = \limsup_{n\to\infty} \frac{S_n - S_k}{\sqrt{n}},$$

so Z is $\sigma(X_{k+1}, X_{k+2}, \ldots)$ measurable. We have thus verified that Z is measurable with respect to the tail σ-algebra, which by Kolmogorov's 0-1 law, is trivial.
Let $M > 0$ (finite). By the central limit theorem,

$$\lim_{n\to\infty} \mathbb{P}[S_n/\sqrt{n} \geqslant M] =: c_M > 0.$$

In particular, for every k,

$$\liminf_{k\to\infty} \mathbb{P}\left[\sup_{n\geqslant k} \frac{S_n}{\sqrt{n}} \geqslant M\right] \geqslant c_M > 0.$$

Actually, the liminf above is a true limit since the events in the probability are decreasing, and we have

$$\mathbb{P}[Z \geqslant M] = \mathbb{P}\left[\limsup_{n\to\infty} \frac{S_n}{\sqrt{n}} \geqslant M\right] = \lim_{k\to\infty} \mathbb{P}\left[\sup_{n\geqslant k} \frac{S_n}{\sqrt{n}} \geqslant M\right] \geqslant c_M > 0.$$

But since Z is measurable with respect to the tail σ-algebra, we must have $\mathbb{P}[Z \geqslant M] \in \{0,1\}$. So $\mathbb{P}[Z \geqslant M] = 1$, and this for every finite M. So $\mathbb{P}[Z = +\infty] = 1$.
(ii) Let $T_n = S_{n!}$, and $T'_n = T_n - T_{n-1}$. By construction, T'_n depends only on $X_{(n-1)!+1}, \ldots, X_{n!}$, so it is independent of T_{n-1}.
Assume by contradiction that $S_n/\sqrt{n}$ converges in probability to some random variable N. Then it must be that N has law $\mathcal{N}(0,\sigma^2)$, with $\sigma = \mathbb{E}[X_1^2]^{1/2} > 0$. T_n being a subsequence of S_n, we also have that $T_n/\sqrt{n!}$ converges in probability to N, and as a consequence,

$$\frac{T_n}{\sqrt{n!}} - \frac{T_{n-1}}{\sqrt{(n-1)!}} \xrightarrow[n\to\infty]{(p)} 0.$$

(Note that we could not do the same reasoning here with convergence in law!) Recall that $T_n = T'_n + T_{n-1}$. Since $T_{n-1}/\sqrt{(n-1)!}$ converges in probability, it must be that $T_{n-1}/\sqrt{n}$ converges to 0 in probability, and thus

$$\frac{T'_n}{\sqrt{n!}} - \frac{T_{n-1}}{\sqrt{(n-1)!}} \xrightarrow[n\to\infty]{(p)} 0. \tag{1}$$

But $T'_n/\sqrt{n!}$ and $T_{n-1}/\sqrt{(n-1)!}$ are independent, and they converge separately to $\mathcal{N}(0,\sigma^2)$. So they jointly converge (see exercise 6 of series 10) to two independent random variables with common law $\mathcal{N}(0,\sigma^2)$. Hence, the difference converges in law to the difference of these two random variables, which has law $\mathcal{N}(0,2\sigma^2)$. But this contradicts (1)!