Series 10: characteristic functions **Solutions**

Exercise 1
$(i.)$ $\varphi_{-X}(\xi) = \mathbb{E}(e^{i\xi\cdot(-X)}) = \mathbb{E}(\cos(\xi \cdot X)) - i\mathbb{E}(\sin(\xi \cdot X)) = \overline{\varphi_X(\xi)}$.
$(ii.)$ X and $-X$ have the same law $\Leftrightarrow$
$\forall \xi \in \mathbb{R}^d,\ \varphi_X(\xi) = \varphi_{-X}(\xi) \Leftrightarrow$
$\forall \xi \in \mathbb{R}^d,\ \varphi_X(\xi) = \overline{\varphi_X(\xi)} \Leftrightarrow$
$\forall \xi \in \mathbb{R}^d,\ \varphi_X(\xi) \in \mathbb{R}$.
$(iii.)$ $\mathbb{E}(e^{i\xi\cdot Z}) = \mathbb{E}\left(\sum_{i=1}^{\kappa} e^{i\xi\cdot X_i} \cdot I_{(\theta=i)}\right) = \sum_{i=1}^{\kappa} \mathbb{E}(e^{i\xi\cdot X_i}) \cdot \mathbb{P}(\theta = i) = \sum_{i=1}^{\kappa} \lambda_i\, \varphi_{X_i}(\xi)$.

Exercise 2
Let X_1, X_2, θ be independent random variables such that:

- X_1 has characteristic function φ;
- X_2 has same law as $-X_1$;
- $\mathbb{P}(\theta = 0) = \mathbb{P}(\theta = 1) = \frac{1}{2}$.

Then, using Exercise 1,

$$\varphi_{X_\theta}(\xi) = \frac{1}{2}\varphi_{X_1}(\xi) + \frac{1}{2}\varphi_{X_2}(\xi) = \frac{1}{2}\varphi(\xi) + \frac{1}{2}\overline{\varphi(\xi)} = \mathrm{Re}\ \varphi(\xi),$$
$$\varphi_{X_1+X_2}(\xi) = \varphi_{X_1}(\xi) \cdot \overline{\varphi_{X_1}(\xi)} = |\varphi(\xi)|^2.$$

Exercise 3
We write μ for the law of X.

$$\begin{aligned} \frac{1}{2T}\int_{-T}^{T} e^{-ita}\varphi(t)\, dt &= \frac{1}{2T}\int_{-T}^{T}\int e^{-it(a-x)}\, d\mu(x)\, dt \\ &= \frac{1}{2T}\int\int_{-T}^{T} e^{-it(a-x)}\, dt\, d\mu(x) \\ &= \int \frac{\sin(T(a-x))}{T(a-x)}\, d\mu(x). \qquad (1) \end{aligned}$$

Define $f_T(x) = \frac{\sin(T(a-x))}{T(a-x)}$. Notice that

$$\forall x, \lim_{T\to\infty} f_T(x) = \mathbb{1}_{\{a\}}(x),$$
$$\sup_{x\in\mathbb{R}, T>0} f_T(x) < \infty.$$

Letting T tend to infinity in (1) and using the dominated convergence theorem, we obtain the desired result since $\int \mathbb{1}_{\{a\}}(x)\, d\mu(x) = \mathbb{P}(X = a)$.

Exercise 4
$(i.)$ Let μ be the law of X. Assume that $\mu\left(\left(\frac{2\pi}{\lambda}\mathbb{Z}\right)^c\right) > 0$. Then,

$$|\mathrm{Re}\ \varphi(\lambda)| = \left|\int \cos(\lambda x)\, d\mu(x)\right| \le \int_{\frac{2\pi}{\lambda}\mathbb{Z}} |\cos(\lambda x)|\, d\mu(x) + \int_{\left(\frac{2\pi}{\lambda}\mathbb{Z}\right)^c} |\cos(\lambda x)|\, d\mu(x).$$

Now, if $x \in \left(\frac{2\pi}{\lambda}\mathbb{Z}\right)^c$, then $|\cos(\lambda x)| < 1$, so this sum is strictly less than

$$\int_{\frac{2\pi}{\lambda}\mathbb{Z}} 1 \, d\mu(x) + \int_{\left(\frac{2\pi}{\lambda}\mathbb{Z}\right)^c} 1 \, d\mu(x) = 1,$$

so $|\text{Re } \varphi(\lambda)| < 1$, so $\varphi(\lambda) \neq 1$.

$(ii.)$We can choose $\alpha, \beta \in (-\delta, \delta)$ such that $\left(\frac{2\pi}{\alpha}\mathbb{Z}\right) \cap \left(\frac{2\pi}{\beta}\mathbb{Z}\right) = \{0\}$ (it suffices to take them with $\alpha/\beta \notin \mathbb{Q}$). Then, $\varphi(\alpha) = \varphi(\beta) = 1$ implies that μ is concentrated on $\left(\frac{2\pi}{\alpha}\mathbb{Z}\right) \cap \left(\frac{2\pi}{\beta}\mathbb{Z}\right) = \{0\}$, so μ is a point mass at zero.

Exercise 5

We will make use of the following two inequalities:

$$\mathbb{P}(|X| > 2/u) \leq \frac{1}{u} \int_{-u}^{u} (1 - \varphi_X(t)) \, dt,$$

$$|\varphi_X(t+h) - \varphi_X(t)| \leq \mathbb{E}|e^{ihX} - 1|.$$

$(i.)$ Suppose that (X_i) is tight and fix $\epsilon > 0$. Using tightness, choose $M > 0$ such that for all $i, \mathbb{P}(|X_i| > M) < \epsilon/4$. Using the continuity of the function $h \mapsto |e^{ih} - 1|$, choose $\delta > 0$ such that $|h| < \delta, |x| < M \implies |e^{ihx} - 1| < \epsilon/2$. Also notice that $|e^{ihM} - 1| \leq 2$ for every $h \in \mathbb{R}$. Putting things together, when s, t are such that $|s - t| < \delta$ and $j \in I$ we get

$$\begin{aligned}
\left|\mathbb{E}(e^{itX_j} - e^{isX_j})\right| &\leq \mathbb{E}\left(|e^{isX_j}| \cdot |e^{i(t-s)X_j} - 1|\right) = \mathbb{E}\left(|e^{i(t-s)X_j} - 1|\right) \\
&= \mathbb{E}\left(|e^{i(t-s)X_j} - 1| \cdot I_{\{|X_j| > M\}}\right) + \mathbb{E}\left(|e^{i(t-s)X_j} - 1| \cdot I_{\{|X_j| \leq M\}}\right) \\
&\leq 2\mathbb{P}(|X_j| > M) + \epsilon/2 = 2\epsilon/4 + \epsilon/2 = \epsilon.
\end{aligned}$$

Now assume that we have the equicontinuity condition of the characteristic functions φ_{X_i} and fix $\epsilon > 0$. Choose $\delta > 0$ such that $|s - t| < \delta \implies |\varphi_{X_j}(s) - \varphi_{X_j}(t)| < \epsilon$ for every $j \in I$. notice that, since $\varphi_{X_j}(0) = 1 \ \forall j$, in particular we have $|\varphi_{X_j}(s) - 1| < \epsilon$ whenever $|s| < \delta$. Now define $M = 2/\delta$; we then have, for all $j \in I$,

$$\mathbb{P}(X_j > M) \leq \frac{1}{\delta} \int_{-\delta}^{\delta} |\varphi_{X_j}(s) - 1| ds \leq \frac{2\epsilon\delta}{\delta} = 2\epsilon,$$

and since ϵ is arbitrary, the proof is complete.

$(ii.)$ See Theorem 3.2.7 of the textbook (or Theorem 2.2.6 in the second edition).

$(iii.)$ It suffices to prove the result for compact sets of the form $[-K, K]$ with $K > 0$, because: $a.)$ any compact set of $\mathbb{R}$ is contained in an interval of this form and $b.)$ if a sequence of functions converges uniformly in a set, it also converges uniformly in any subset of this set. So, from now on, fix $K > 0$ and $\epsilon > 0$, and we want to show that there exists N such that, if $n \geq N$, then $|\varphi_{X_n}(x) - \varphi_\mu(x)| < \epsilon$ for any $x \in [-K, K]$.

Let μ_n denote the distribution of X_n. By $(ii.)$, the set of distributions $\{\mu\} \cup \{\mu_n : n \in \mathbb{N}\}$ is tight. By part $(i.)$, we can thus choose $\delta > 0$ such that, whenever $|s - t| < \delta$, we have $|\varphi_{\mu_n}(s) - \varphi_{\mu_n}(t)| < \epsilon/3$ and $|\varphi_\mu(s) - \varphi_\mu(t)| < \epsilon/3$. Now take $x_0, x_1, \ldots, x_k$ such that $x_0 = -K, x_k = K$ and $0 < x_{i+1} - x_i < \delta$ for each i. Finally, choose N such that $n \geq N$ implies $|\varphi_{\mu_n}(x_i) - \varphi_\mu(x_i)| < \epsilon/3$ for every i. Now, fix $n \geq N$ and $x \in [-K, K]$. There exists i such that $x - x_i < \delta$; we then have

$$|\varphi_{\mu_n}(x) - \varphi_\mu(x)| \leq |\varphi_{\mu_n}(x) - \varphi_{\mu_n}(x_i)| + |\varphi_{\mu_n}(x_i) - \varphi_\mu(x_i)| + |\varphi_\mu(x_i) - \varphi_\mu(x)| < 3\epsilon/3 = \epsilon.$$

Exercise 6

(i.) $X_n + Y_n$ converges in distribution to a law μ if and only if $\forall t$, $\varphi_{X_n+Y_n}(t)$ converges to $\varphi_\mu(t)$. Notice that $\varphi_{X_n+Y_n}(t) = \varphi_{X_n}(t)\,\varphi_{Y_n}(t) \to \varphi_{X_\infty}(t)\,\varphi_{Y_\infty}(t) = \varphi_{X_\infty+Y_\infty}(t)$, where $\varphi_{X_\infty+Y_\infty}$ is the characteristic function associated to the law of a sum of a random variable with the law of X_∞ with another random variable with the law of Y_∞, the two being independent. This concludes the proof.

(ii.) Part of the statement is that the function $\prod_{i=1}^\infty \varphi_{X_i}(t)$, equal to $\lim_{N\to\infty} \prod_{i=1}^N \varphi_{X_i}(t)$, is well-defined, i.e. that the limit exists for all t. Indeed, on the one hand we have $\varphi_{X_1+\ldots+X_N}(t) = \prod_{i=1}^N \varphi_{X_i}(t)$, and on the other hand the fact that $X_1 \ldots + X_N$ converges almost surely to S_∞ implies that $X_1 + \ldots + X_N \Rightarrow S_\infty$, so $\varphi_{X_1+\ldots+X_N}(t) \to \varphi_{S_\infty}(t)$ for all t. This shows that $\varphi_{S_\infty}(t) = \lim_{N\to\infty} \varphi_{X_1+\ldots+X_N}(t) = \lim_{N\to\infty} \prod_{i=1}^N \varphi_{X_i}(t) = \prod_{i=1}^\infty \varphi_{X_i}(t)$.