Solutions to Practice Test

November 8, 2013

1) The sigma algebra generated by the functions $\omega \to \omega_t \;\; t \geq 0$ is the smallest sigma algebra such that al the maps $\omega \to \omega_t$ are measureable with respect to it. Equally it is the smallest sigma algebra which contains all the sets

$$\{\omega : \omega_t \leq c\}$$

for all t and c. We denote this sigma algebra by sigma algegra $\mathcal{G}$ Let $\mathcal{G}'$ be the collection of sets of the form

$$A \;=\; \{\omega : \omega_{t_1}, \omega_{t_2}, \cdots \omega_{t_n}, \cdots \in H\}$$

for some sequence $t_i \geq 0$ and some H in the Borels of $\mathbb{R}^N, \mathcal{H}$. Now (Fact but can be cited) the Borels on $\mathbb{R}^N$ are generated by the sets of the form

$$\mathbb{R} \times \mathbb{R} \times \cdots \mathbb{R} \times O \times \mathbb{R} \times \mathbb{R} \times \cdots$$

. Thus if we consider $\mathcal{H}'$ to be the collection of subsets of $\mathbb{R}^N$ having the property that for each sequence $t_i \geq 0$, the set

$$A \;=\; \{\omega : \omega_{t_1}, \omega_{t_2}, \cdots \omega_{t_n}, \cdots \in H\}$$

is a member of $\mathcal{G}$, we find that

1) $\mathbb{R}^N$ is in $\mathcal{H}'$ as in this case whatever the choice of sequence $t_i \geq 0$, the set A is equal to $\mathbb{R}^{\mathbb{R}_+}$,

2) if $H \in \mathcal{H}'$, then $H^c \in \mathcal{H}'$. Since for any sequence $t_i \geq 0$, if $A \;=\; \{\omega : \omega_{t_1}, \omega_{t_2}, \cdots \omega_{t_n}, \cdots \in H\}$is in $\mathcal{G}$, then $A^c \;=\; \{\omega : \omega_{t_1}, \omega_{t_2}, \cdots \omega_{t_n}, \cdots \in H^c\}$is in $\mathcal{G}$

3) if $H_i \;\in\; \mathcal{H}' \;\; i \;\; geq1$, then $\cup_i H_i \;\in\; \mathcal{H}'$: for any sequence t_i we have $A_i \;=\; \{\omega : \omega_{t_1}, \omega_{t_2}, \cdots \omega_{t_n}, \cdots \in H_i\}$is in $\mathcal{G}$, so $\cup A_i$ is in $\mathcal{G}$ but

$$\cup A_i \;=\; \{\omega : \omega_{t_1}, \omega_{t_2}, \cdots \omega_{t_n}, \cdots \in \cup H_i\}$$

and so $\cup H_i$ is in $\mathcal{H}'$.

Thus $\mathcal{H}'$ is a sigma field which contains elements $\mathbb{R} \times \mathbb{R} \times \cdots \mathbb{R} \times O \times \mathbb{R} \times \mathbb{R} \times \cdots$ for O Borellian on $\mathbb{R}$ so it contains $\mathcal{H}$. Thus we can conclude that $\mathcal{G}' \;\subset\; \mathcal{G}$.

For the converse inclusion we start by noting that the above argument gives that for any sequence (not necessarily increasing) of positive integers $n_1, n_2, \cdots$ we have for every $H \in \mathcal{H}$, the set

$$\{\omega \in \mathbb{R}^N : \omega_{n_1}, \omega_{n_2} \cdots \in H\}$$

is in $\mathcal{H}$.

We consider $\mathcal{G}'$. Since it contains sets of the form $\{\omega_t \in O\}$ for O a Borellian subset of $\mathbb{R}$, to show that $\mathcal{G} \subset \mathcal{G}'$ it is enough to show that $\mathcal{G}'$ is a sigma field.

Firstly, the fact that $\mathbb{R}^{\mathbb{R}_+} \in \mathcal{G}'$ follows immediately by choosing H to be $\mathbb{R}^N$ and the sequence t_i to be arbitrary.
Secondly, again if $A \in \mathcal{G}'$, then for H the corresponding set in $\mathbb{R}^N$ and t_i the given sequence, we have

$$A^c = \{\omega : \omega_{t_1}, \omega_{t_2}, \cdots \omega_{t_n}, \cdots \in H^c\}$$

which is evidently in $\mathcal{G}'$ since H^c is in $\mathcal{H}$.

It remains to deal with countable unions. Suppose that A_i, $i = 1, 2 \cdots$ is a sequence of elements of $\mathcal{G}'$ and suppose that

$$A_i = \{\omega : \omega_{t_1^i}, \omega_{t_2^i}, \cdots \omega_{t_n^i}, \cdots \in H_i\}.$$

Since the countable union of countable subsets is countable, we may write

$$\cup_i \cup_j \{t_j^i\}$$

as $t_1, t_2, \cdots$. Let n_j^i be such that $t_j^i = t_{n_j^i}$. Then

$$\cup_i A_i = \{\omega : \omega_{t_1}, \omega_{t_2}, \cdots \omega_n, \cdots \in H\}$$

where H is equal to the set

$$\{\omega \in \mathbb{R}^N : \text{ for some } i, \omega_{n_1^i}, \omega_{n_2^i} \cdots \in H_i\} = \cup_i H_i'$$

for $H_i' = \{\omega : \omega_{n_1^i}, \omega_{n_2^i}, \cdots \omega_{n_n^i}, \cdots \in H_i\}$. Thus $\cup A_i \in \mathcal{G}'$ and we are done.

2) Use Fatous Lemma on $X_n = 1 - \mathbb{1}(A_n)$ to get

$$1 - E(\limsup_n \mathbb{1}(A_n)) = E(\liminf_n X_n) \leq \liminf_n E(X_n) = 1 - \limsup_n P(A_n).$$

Note that $X_n \geq 0$ $a.s.$ and thus Fatous lemma is applicable. Also,

$$\limsup_n \mathbb{1}(A_n) = \mathbb{1}(\limsup_n A_n)$$

since $\omega \in \limsup_n A_n$ if and only if ω belongs to infinitely many A_n if and only if $\limsup_n \mathbb{1}(A_n)(\omega) = 1$. This proves the first part.

For second part, set U_n to be i.i.d. Uniform $[0,1]$. And set $A_n = \{U_n \leq 0.5\}$. We know that $P(A_n) = 0.5$ for all $n \geq 1$. Thus we have that

$$\sum_n P(A_n) = \infty$$

and hence by Borel-Cantelli Lemma $P(\limsup_n A_n) = 1$.

3) Set $A_n = \{U_{2n} > 2nU_{2n+1}\}$ and $B_n = \{U_{2n-1} > (2n-1)U_{2n}\}$. We then have that

$$P(A_n) = \int_0^1 P(U_{2n} > 2nx)dx = \int_0^{(2n)^{-1}} P(U_{2n} > x)dx = \int_0^{(2n)^{-1}} (1-x)dx = \frac{1}{2n} - \frac{1}{8n^2}.$$

This implies that $\sum_n P(A_n) = \infty$. Since A_n are independent, we have by Borel-Cantelli II lemma that

$$P(\limsup_n A_n) = 1$$

and by Corollary 4.5, we have

$$\frac{\sum_{i=1}^n \mathbb{1}(A_i)}{\sum_{i=1}^n P(A_i)} \longrightarrow 1 \ \ a.s.$$

as $n \to \infty$. Similarly,

$$P(\limsup_n B_n) = 1$$

and

$$\frac{\sum_{i=1}^n \mathbb{1}(B_i)}{\sum_{i=1}^n P(B_i)} \longrightarrow 1 \ \ a.s.$$

as $n \to \infty$. Since $P(A_i) = P(B_i)$ for each i, we then have that

$$\frac{\sum_{i=1}^n \mathbb{1}(B_i) + \mathbb{1}(A_i)}{\sum_{i=1}^n P(B_i) + P(A_i)} \longrightarrow 1 \ \ a.s.$$

as $n \to \infty$. In other words,

$$\frac{\sum_{i=1}^n \mathbb{1}(U_i > iU_{i+1})}{\sum_{i=1}^n P(U_i > iU_{i+1})} \longrightarrow 1 \ \ a.s.$$

as $n \to \infty$.

4) We use the following weak law for triangular arrays (5.5) of Durrett (1996) with a computation analogous to St. Petersburg Paradox (Example 5.7). We first note that

$$P(X_1 \geq x) = \int_x^\infty y^{-2}dy = \frac{1}{x} \tag{1}$$

for all $x \geq 1$.

For $n \geq 1$, let $m(n) = \log n + K(n)$, where $K(n) \to \infty$ and is chosen so that $m(n)$ is an integer. Letting $b_n = e^{m(n)}$ we define

$$X_{n,k} := X_k \mathbb{1}(|X_k| \leq b_n) = X_k \mathbb{1}(X_k \leq b_n).$$

We know from (5.5) of Durrett (1996) that if:
(i) $\sum_{k=1}^{n} P(|X_{n,k}| > b_n) \to 0$ and
(ii) $b_n^{-2} \sum_{k=1}^{n} EX_{n,k}^2 \to 0$ as $n \to \infty$, then

$$\frac{S_n - a_n}{b_n} \to 0 \tag{2}$$

in probability where $S_n = X_1 + ... + X_n$ and $a_n = \sum_{k=1}^{n} EX_{n,k}$.

We use the above for our proof. To check (i) holds, first we note from (1) that

$$P(X_1 \geq e^m) = \frac{1}{e^m}$$

for $m \geq 1$. Thus we have

$$\sum_{k=1}^{n} P(|X_{n,k}| > b_n) = \sum_{k=1}^{n} P(X_{n,k} > b_n) = nP(X_1 \geq b_n) = nb_n^{-1} = e^{-K(n)} \longrightarrow 0$$

as $n \to \infty$, since $K(n) \to \infty$. This proves that (i) holds.

To check (ii), we observe that

$$EX_{n,k}^2 = \int_1^{b_n} dx = b_n.$$

So

$$b_n^{-2} \sum_{k=1}^{n} EX_{n,k}^2 = \frac{n}{b_n} = e^{-K(n)}$$

which converges to zero as $n \to \infty$ since $K(n) \to \infty$.

The last step is to evaluate a_n. We know that

$$E(X_{n,k}) = \int_1^{b_n} \frac{dx}{x} = m(n)$$

so that $a_n = nm(n)$. We have $m(n) = \log n + K(n)$, so if we pick

$$\frac{K(n)}{\log n} \to 0$$

then

$$\frac{a_n}{n \log n} \longrightarrow 1 \tag{3}$$

as $n \to \infty$. If we pick

$$K(n) = \sup\{k \leq \log\log n : \log n + k \quad \textit{is an integer}\}$$

then for all large n, we must have $b_n = e^{m(n)} \leq n \log n$ and (2) implies that

$$\frac{|S_n - a_n|}{n \log n} \leq \frac{|S_n - a_n|}{b_n} \longrightarrow 0$$

in probability. From (3) we get that

$$\frac{S_n}{n \log n} \longrightarrow 1$$

in probability as $n \to \infty$.

To show that a.s. convergence does not hold, we note that

$$P(X_n > 2n \log n) = \frac{1}{2n \log n}.$$

(The number 2 is not important here. Instead of 2, we can also choose any number larger than one.) By Borel-Cantelli Lemma, we have that

$$P(A_n \ \ i.o.) = 1$$

where $A_n = \{X_n > 2n \log n\}$. But if A_n occurs then

$$S_n > X_n > 2n \log n$$

so that we have

$$P(B_n \ \ i.o.) = 1$$

where $B_n = \frac{S_n}{n \log n} > 2$.

5) Let $\omega \in [0, 1]$. We note that $X_k(\omega) \in \{0, 1\}$ for all $k \geq 1$ and for any $i_1, ..., i_n \in \{0, 1\}$, we have

$$\{\omega : X_1(\omega) = i_1, ..., X_n(\omega) = i_n\} = \{\omega \in I_{n,k}\} \tag{4}$$

where $I_{n,k} = [\sum_{k=1}^n i_k 2^{-k}, \sum_{k=1}^n i_k 2^{-k} + 2^{-n})$.

Thus we have

$$0 \leq \omega - \sum_{k=1}^n X_k(\omega) 2^{-k} \leq \frac{1}{2^n}$$

for all $n \geq 1$. Indeed, for $n = 1$, we have $X_1(\omega) = [2\omega] \ mod \ 2$. If $X_1(\omega) = 1$, then $\omega \geq 0.5$ and hence

$$0 \leq \omega - X_1(\omega)0.5 = \omega - 0.5 \leq 0.5$$

If on the other hand, $X_1(\omega) = 0$, then $\omega < 0.5$ and again

$$0 \leq \omega - X_1(\omega)0.5 = \omega < 0.5.$$

An analogous procedure for general n holds.

The $\{X_i\}_i$ are discrete random variables. Therefore to show independence, it suffices to show that

$$P(X_1 = i_1, ..., X_n = i_n) = \prod_{k=1}^{n} P(X_k = i_k),$$

for any $i_1, ..., i_n \in \{0, 1\}$. First, so that

$$P(X_1 = i_1, ..., X_n = i_n) = \frac{1}{2^n}.$$

Summing over all $i_1, ..., i_{n-1} \in \{0, 1\}$, we thus get

$$P(X_n = i_n) = 0.5$$

for $i_n \in \{0, 1\}$. This shows that X_n is distributed as a Bernoulli random variable with parameter 0.5. Thus

$$P(X_1 = i_1, ..., X_n = i_n) = 2^{-n} = \prod_{k=1}^{n} P(X_k = i_k)$$

and we have that $\{X_k\}_k$ are independent random variables.

Finally, the random variables U_1 and U_2 are independent (Corollary 4.5 of Durrett (1996)). To deduce that U_1 is uniformly distributed in $[0, 1]$, one way to proceed is as follows. Get

$$P(U_1 \leq \sum_{k=1}^{n} i_k 2^{-n}) = \sum_{k=1}^{n} i_k 2^{-k}.$$

And using right continuity get that

$$P(U_1 \leq t) = t$$

for all t.