
Series 7: convergence in probability, strong law of large numbers

Exercise 1

If $X_n \rightarrow X$ in probability and $X_n \rightarrow Y$ in probability, then $X = Y$ a.s. ?

Exercise 2

Let $(X_n)_{n \geq 1}$ be a sequence of random variables, and Z be a random variable. Show that, if for any subsequence $(n_k)_{k \geq 1}$, there exists a sub-subsequence $(n_{k_\ell})_{\ell \geq 1}$ such that $X_{n_{k_\ell}} \xrightarrow{P} Z$, then $X_n \xrightarrow{P} Z$.

Exercise 3

Show that if $X_n \xrightarrow{P} X$, then $f(X_n) \xrightarrow{P} f(X)$ for any continuous function $f: \mathbb{R} \rightarrow \mathbb{R}$.

Exercise 4

Show that for any sequence of random variables $(X_n)_{n \geq 1}$, we have

$$\frac{S_n}{n} \xrightarrow{P} 0 \implies \frac{X_n}{n} \xrightarrow{P} 0,$$

where $S_n = \sum_{i=1}^n X_i$.

Exercise 5

Let $X_1, X_2, \dots$ be independent random variables with distribution $N(0, 1)$. What is the almost sure limit of

$$\frac{X_1^2 + \dots + X_n^2}{(X_1 - 1)^2 + \dots + (X_n - 1)^2} ?$$

Exercise 6

Let $(X_i)_{i \geq 1}$ be a sequence of i.i.d. random variables with density

$$f(x) = \begin{cases} e^{-(x+1/2)}, & x \geq -1/2, \\ 0, & x < -1/2. \end{cases}$$

Show that if $S_n = \sum_{j=1}^n X_j$, then $S_n \rightarrow +\infty$ almost surely.

Exercise 7

Let $(X_i)_{i \geq 1}$ be a sequence of i.i.d. random variables such that $X_1 \sim U[0, 1]$. What is the almost sure limit of

$$\left(\prod_{i=1}^n X_i \right)^{1/n} ?$$

Exercise 8

Give an example of two sequences (X_n) and (N_n) of random variables such that:

- $X_n(\omega) \in \{0, 1\} \forall \omega, \forall n$;
- $N_n(\omega) \in \mathbb{N} \forall \omega, \forall n$;
- $1 \leq N_n(\omega) < N_{n+1}(\omega) \forall \omega, \forall n$;
- $X_n \rightarrow 0$ in probability;
- $X_{N_n} \rightarrow 1$ almost surely.

Exercise 9

Let $(X_k)_{k \geq 1}$ be a sequence of independent random variables distributed uniformly on $\{1, 2, \dots, n\}$. Let

$$T_n = \inf \{m : \{X_1, \dots, X_m\} = \{1, \dots, n\}\}$$

be the first instant when all possible values have been observed. You may think of the integers $\{1, 2, \dots, n\}$ as coupons that you want to collect and that you obtain at random, and T_n is the time when your collection is complete. The aim is to show that $T_n/(n \log(n))$ tends to 1 in probability.

1. Let $\tau_k^n = \inf\{m : |\{X_1, \dots, X_m\}| = k\}$ be the first instant when k different items have been obtained. Show that for $k \leq n - 1$, the random variables $(\tau_{k+1}^n - \tau_k^n)$ are independent, and are distributed respectively according to a geometric law of parameter $1 - k/n$.
2. Prove the claim.