Series 3: random variables and expectation

Exercise 1
Let $C = \{A_1, A_2, \ldots, A_\kappa\}$ be a partition of $\Omega = \bigcup_{i=1}^{\kappa} A_i$. Assume that $Y : \Omega \to \mathbb{R}$ is $\sigma(C)$-measurable, i.e. $Y^{-1}(E) \in \sigma(C)$, $\forall E \in \mathcal{B}(\mathbb{R})$. Show that Y is constant on A_i for any $i \in \{1, 2, \ldots \kappa\}$.

Exercise 2
Let X be a real random variable on $(\Omega, \mathcal{F})$, and $\sigma(X)$ be the σ-algebra generated by X, that is,

$$\sigma(X) = \left\{X^{-1}(E) : E \in \mathcal{B}(\mathbb{R})\right\}.$$

(a) Show that if $Y = f \circ X$, where $f : \mathbb{R} \to \mathbb{R}$ is measurable, then Y is $\sigma(X)$-measurable, i.e. $Y^{-1}(E) \in \sigma(X)$, $\forall E \in \mathcal{B}(\mathbb{R})$.

(b) Show that if Y is $\sigma(X)$-measurable, then there exists a measurable function $f : \mathbb{R} \to \mathbb{R}$ such that $Y = f \circ X$. *(Hint: assume first that Y is a simple function. For the general case, use the fact that Y is the pointwise limit of a sequence of simple functions.)*

Exercise 3
With any random variable X with values in $\mathbb{R}^d$, we associate its cumulative distribution function $F_X : \mathbb{R}^d \to [0, 1]$ defined by

$$F_X(x) = \mathbb{P}[X \leqslant x] \quad (= \mathbb{P}[\forall i \in \{1, \ldots d\}, X_i \leqslant x_i]).$$

Show that two random variables with values in $\mathbb{R}^d$ have the same cumulative distribution function if and only if they have the same law.

Exercise 4
Let X be a real random variable. Show that $\mathbb{E}[X^2] = (\mathbb{E}[X])^2 < +\infty$ if and only if X is constant almost surely, i.e. there exists a constant $c \in \mathbb{R}$ such that $\mathbb{P}[X = c] = 1$.

Exercise 5
Let X be an integrable real random variable defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

(a) Show that if $A_1, A_2, A_3, \ldots$ is a sequence of measurable subsets of Ω such that for all n, $A_n \subseteq A_{n+1}$, then

$$\lim_{n\to\infty} \int_{A_n} |X| \, d\mathbb{P} = \int_{\cup_{n=1}^{\infty} A_n} |X| \, d\mathbb{P}$$

(b) Show that there exists a sequence of random variables $(X_n)_{n \geqslant 1}$ such that for all n, $|X_n| \leqslant \log(n)$, and moreover, $\lim_{n\to+\infty} \mathbb{E}[X_n] = \mathbb{E}[X]$.

Exercise 6
Let μ be a measure on the set E, and $f_n : E \to \mathbb{R}_+$ be a sequence of positive measurable functions that converge μ-almost everywhere to a function f. We assume that $\int f_n \, d\mu \xrightarrow[n\to+\infty]{} \int f \, d\mu$. Is it true that $\int |f_n - f| \, d\mu \xrightarrow[n\to+\infty]{} 0$?