Series 2: random variables

Exercise 1
Let X and Y be two random variables on $(\Omega, \mathcal{F}, \mathbb{P})$, and $A \in \mathcal{F}$. Show that if we define $Z(\omega) = X(\omega)$ for $\omega \in A$ and $Z(\omega) = Y(\omega)$ for $\omega \in A^c$, then Z is also a random variable.

Exercise 2
Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Show that $E = \{\omega \in \Omega : \mathbb{P}(\{\omega\}) > 0\}$ is a countable subset of Ω. Using this result, show that a cumulative distribution function has at most countably many discontinuities.

Exercise 3
Show that $X = (X_1, X_2, \ldots, X_n)$ is a random variable with values in $\mathbb{R}^n$ if and only if for any $i \in \{1, \ldots, n\}$, X_i is a random variable.

Exercise 4
(1) Consider $\mathbb{R}/\mathbb{Q}$ the set of equivalence classes of the real numbers modulo the rational numbers (in other words, two real numbers are identified if their difference is rational). For each $a \in \mathbb{R}/\mathbb{Q}$, we choose a representative of the class $x_a \in [0, 1]$. Let

$$A = \{x_a,\ a \in \mathbb{R}/\mathbb{Q}\}.$$

Show that A is not Borel measurable (i.e. $A \notin \mathcal{B}(\mathbb{R})$).
(2) Let $(X_\alpha)_{\alpha \in I}$ be a collection of real random variables on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and let

$$X = \sup_{\alpha \in I} X_\alpha.$$

Is X always a random variable ?

Exercise 5
Let $(\Omega, \mathcal{F}, P)$ be a probability space. Show that if $A_1, A_2, \ldots, A_n \in \mathcal{F}$ then

$$P\left(\cup_{i=1}^n A_i\right) = \sum_{i=1}^n P(A_i) - \sum_{i<j} P(A_i \cap A_j) + \sum_{i<j<k} P(A_i \cap A_j \cap A_k) - \cdots + (-1)^{n-1} P(\cap_{i=1}^n A_i)\,.$$