Series 11: characteristic functions and central limit theorem

Exercise 1
(i.) Show that if X_n is a sequence of random variables that converge in law to $\mathcal{N}(0,1)$, and $(a_n)_{n\geq 1}$ is a real sequence such that $a_n \to a$, then $X_n + a_n \xrightarrow{(\mathcal{L})} \mathcal{N}(a,1)$.
(ii.) Show that if $X_n \sim \mathcal{N}(0,\sigma_n^2)$ converges in law to X, then $\sigma_n^2 \to \sigma \in [0,\infty)$.

Exercise 2
Show that if $X \sim \mathcal{N}(0,1)$, then

$$\mathbb{E}\left[X^{2n}\right] = \frac{(2n)!}{2^n n!}.$$

Exercise 3
Use the central limit theorem for Poisson random variables to show that

$$\lim_{n\to\infty} e^{-n} \sum_{k=0}^{n} \frac{n^k}{k!} = \frac{1}{2}.$$

Exercise 4
Let $X_1, X_2, \ldots$ be i.i.d. real random variables with $\mathbb{E}[X_1] = 1$, $\mathrm{Var}(X_1) = \sigma^2 \in (0,\infty)$. Show that

$$\sqrt{S_n} - \sqrt{n} \xrightarrow[n\to\infty]{(\mathcal{L})} \mathcal{N}(0,\sigma^2/4).$$

Exercise 5
Let $X_1, X_2, \ldots$ be i.i.d. real random variables with $\mathbb{E}[X_1] = 0$, $0 < \mathbb{E}[X_1^2] < \infty$. Show that

$$\left(\sum_{k=1}^{n} X_k^2\right)^{-1/2} \sum_{k=1}^{n} X_k \xrightarrow[n\to\infty]{(\mathcal{L})} \mathcal{N}(0,1).$$

Exercise 6
Let $X_1, X_2, \ldots$ be i.i.d. real random variables with $\mathbb{E}[X_1] = 0$, $0 < \mathbb{E}[X_1^2] < \infty$, and let $S_n = (X_1 + \ldots + X_n)$.
(i) Show that $\limsup S_n/\sqrt{n} = +\infty$ a.s. (*Hint:* use Kolmogorov's 0-1 law.)
(ii) Show that $S_n/\sqrt{n}$ does not converge in probability. (*Hint:* intuitively, if $n_1 \ll n_2$, then S_{n_1} and S_{n_2} are almost independent.)