Series 10: characteristic functions

Exercise 1
Let X be a random variable with values in $\mathbb{R}^d$, and φ be its characteristic function.
(i.) Show that $\overline{\varphi}$ is the characteristic function of $-X$.
(ii.) Show that $\forall \xi \in \mathbb{R}^d$, $\varphi(\xi) \in \mathbb{R}$ if and only if X and $-X$ have the same distribution.
(iii.) Let $X_1, X_2, \ldots, X_\kappa$ and θ be independent random variables. Assume that

$$P[\,\theta = j\,] = \lambda_j \quad \text{for } j = 1, 2, \ldots, \kappa, \quad \text{where} \quad \sum_{j=1}^{\kappa} \lambda_j = 1\,.$$

Show that the characteristic function φ_Z of $Z = X_\theta$ is

$$\varphi_Z = \lambda_1\,\varphi_{X_1} + \lambda_2\,\varphi_{X_2} + \ldots + \lambda_\kappa\,\varphi_{X_\kappa}\,.$$

Exercise 2
Show that if φ is the characteristic function of a real random variable, then $\mathrm{Re}(\varphi)$ and $|\varphi|^2$ are also characteristic functions.

Exercise 3
Let X be a real random variable, and φ its characteristic function. Show that for every $a \in \mathbb{R}$,

$$\mathbb{P}(X = a) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} e^{-ita}\,\varphi(t)\,dt.$$

Exercise 4
(i.) Let φ be the characteristic function of the real random variable X. Show that if there exists $\lambda \neq 0$ such that $\varphi(\lambda) = 1$, then a.s., X takes values in the set $\frac{2\pi}{\lambda}\mathbb{Z}$.
(ii.) Show that if there exists $\delta > 0$ such that $\varphi(t) = 1$ for every $t \in (-\delta, \delta)$, then actually $\varphi(t) = 1$ for every $t \in \mathbb{R}$. What is the law of X?

Exercise 5
Let $(X_i)_{i\in I}$ be a family of real random variables. (i.) Show that (X_i) is tight if and only if

$$\forall \epsilon > 0, \exists \delta > 0 \text{ s.t. } |s-t| < \delta \implies \forall i \in I,\ |\varphi_{X_i}(s) - \varphi_{X_i}(t)| < \epsilon.$$

(ii.) We now assume $I = \mathbb{N}$, and write μ_n for the law of X_n. Show that if $\mu_n \to \mu$ weakly, then the set of distributions $\{\mu\} \cup \{\mu_n : n \in \mathbb{N}\}$ is tight.
(iii.) If $I = \mathbb{N}$ and there exists μ such that $X_n \to \mu$ in distribution, show that $\varphi_{X_n} \to \varphi$ uniformly over every compact of $\mathbb{R}$.

Exercise 6
(i.) Show that if $X_n \to X_\infty, Y_n \to Y_\infty$ in distribution, and, for all $n \in \mathbb{N} \cup \{\infty\}$, X_n is independent of Y_n, then $X_n + Y_n \to X_\infty + Y_\infty$ in distribution (all random variables taking real values).
(ii.) Show that if $X_1, X_2, \ldots$ are independent and $\sum^\infty X_i = S_\infty$ almost surely, then $\varphi_{S_\infty} = \prod_{i=1}^\infty \varphi_{X_i}$.