Series 8: almost sure convergence

Exercise 1

At time 0, a light bulb is turned on. It breaks down after a time X_1, and then it is replaced by a new light bulb after a time Y_1. The new light bulb works during a time X_2, and once it is broken, the time until it is changed by a new one is Y_2, and so on. We assume that $(X_n, Y_n)_{n\in\mathbb{N}}$ are independent random variables with finite mean. Let R_t be the amount of time in $[0, t]$ when a light bulb was working. Show that

$$\frac{R_t}{t} \xrightarrow[t\to\infty]{\text{a.s.}} \frac{\mathbb{E}[X_1]}{\mathbb{E}[X_1] + \mathbb{E}[Y_1]}.$$

Exercise 2

Let $X_0 = (1, 0)$. We define the sequence of random variables $(X_n)_{n\geqslant 1}$ by induction in the following way: conditionally on $X_0, \dots, X_n$, the variable X_{n+1} is chosen uniformly at random in the disk with center 0 and radius $|X_n|$. Show that $\ln(|X_n|)/n$ converges almost surely to a constant.

Exercise 3

Let $X_0, X_1, \dots$ be i.i.d. random variables, with X_0 not almost surely equal to 0. Show that the radius of convergence of the series $\sum X_n z^n$ is equal either to 1 a.s. or to 0 a.s., depending on whether $\log(|X_0|)\mathbb{1}_{\{|X_0|\geqslant 1\}}$ is integrable or not.

Exercise 4

Let $X_1, X_2, \dots$ be independent real random variables. We let $S_{m,n} = \sum_{k=m}^{n-1} X_k$. Show that

$$P[\max_{m\leqslant j<n} |S_{m,j}| > 2a] \min_{m\leqslant k<n} P[|S_{k,n}| \leqslant a] \leqslant P[|S_{m,n}| > a].$$

Deduce that if $S_{0,n}$ converges in probability as n tends to infinity, then the convergence holds almost surely.