Series 5: weak law of large numbers

Exercise 1
Let $(X_n)_{n\geqslant 1}$ be a sequence of random variables. Show that if $E[X_n] \to \alpha$ and $Var(X_n) \to 0$, then $X_n \xrightarrow{\text{P}} \alpha$.

Exercise 2
Let $X_1, X_2, \ldots$ be random variables. Suppose that $\mathbb{E}X_n = 0$ and $\mathbb{E}X_n X_m \leq r(n-m)$ if $m \leq n$, where $r : \mathbb{N} \to \mathbb{R}$ satisfies $\lim_{k\to\infty} r(k) = 0$. Show that $(X_1 + \ldots + X_n)/n \to 0$ in probability.

Exercise 3
Let $Y_1, Y_2, \ldots$ be i.i.d. random variables. Show that
$(i.)$ $Y_n/n \to 0$ in probability.
$(ii.)$ $Y_n/n \to 0$ a.s. if and only if $\mathbb{E}(|Y_1|) < \infty$.

Exercise 4
Let $X_1, X_2, \ldots$ be independent exponentially distributed random variables, $\lambda_n = \mathbb{E}[X_n]$, and $S_n = X_1 + \cdots + X_n$. Show that if λ_n is uniformly bounded and $\sum \lambda_n$ is infinite, then $S_n/\mathbb{E}[S_n]$ converges to 1 a.s.

Exercise 5
Let $p_k = (2^k k(k+1))^{-1}, k \geq 1$ and $p_0 = 1 - \sum_{k\geq 1} p_k$.

a) Show that

$$\sum_{k=1}^{\infty} 2^k p_k = (1 - \frac{1}{2}) + (\frac{1}{2} - \frac{1}{3}) + \cdots = 1$$

and that if we consider $(X_n\colon n \geq 1)$ i.i.d. such that $\mathbb{P}(X_n = -1) = p_0$, $\mathbb{P}(X_n = 2^k - 1) = p_k$ for$k \geq 1$,then $\mathbb{E}(X_n) = 0$.

b) Show that $S_n/(n/log_2(n)) = \sum_{i=1}^{n} X_i/(n/\log_2(n)) \to -1$ in probability.

Hint : We can use the law of large numbers for triangular arrays [**?**, (5.5), p41] with $b_n = 2^{m(n)}$ and $m(n) = \min\{m\colon 2^{-m}m^{-3/2} \leq n^{-1}\}$.

Exercise 6
Let $(X_n\colon n \geq 1)$ be i.i.d. random variables such that $\mathbb{P}(X_i = (-1)^k k) = C(k^2 \log(k))^{-1}$ for $k \geq 2$ and C a constant ofnormalization. Show that $\mathbb{E}(|X_i|) = \infty$ but that there exists $\mu < \infty$ such that $S_n/n = \sum_{i=1}^{n} X_i/n \to \mu$ in probability as $n \to \infty$.

Exercise 7
Let $X_1, X_2, \ldots$ be i.i.d. real random variables whose common distribution has a density. We say that a record occurs at time k if $X_k > \max_{j<k} X_j$. Let R_n be the number of records that have occurred up to time n. Show that

$$\frac{R_n}{\log n} \xrightarrow[n\to\infty]{\text{a.s.}} 1.$$