Series 1: probability spaces

Exercise 1
Let X: $(\Omega, \mathcal{F}, \mathbb{P}) \to (E, \mathcal{E})$ be a random variable. Recall the definition of the distribution of X, and show that it is indeed a probability measure.

Exercise 2
Let $\Omega = \{1, 2, 3, 4\}$. Find two $\sigma-$algebras $\mathcal{F}_1$ and $\mathcal{F}_2$ of Ω such that the union $\mathcal{F}_1 \cup \mathcal{F}_2$ is not a $\sigma-$algebra.

Exercise 3
Let $\mathcal{F}$ be the collection of all subsets of $\{1, 2, 3, 4\}$. Give an example of two probability measures $\mu \neq \nu$ on $\mathcal{F}$ which agree on a collection of sets C such that $\sigma(C) = \mathcal{F}$.

Exercise 4
A σ-algebra $\mathcal{F}$ is said to be countably generated if there exists a countable collection $C \subseteq \mathcal{F}$ such that $\sigma(C) = \mathcal{F}$.

(a) Show that $\mathcal{B}(\mathbb{R})$ is countably generated.

(b) Show that $\mathcal{B}(\mathbb{R}^2) = \mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R})$.

Exercise 5
Let $\Omega = \mathbb{R}$, and let $\mathcal{F}$ be the collection of all subsets of $\mathbb{R}$ such that either A or A^c is countable. We define $\mathbb{P} : \mathcal{F} \to [0, 1]$ such that $\mathbb{P}(A) = 0$ if A is countable, and $\mathbb{P}(A) = 1$ otherwise. Show that $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space.

Exercise 6
We say that $A \subseteq \mathbb{N}$ has asymptotic density θ if

$$\lim_{n\to\infty} \frac{\#(A \cap \{1, \ldots, n\})}{n} = \theta(A)\,.$$

Let $\mathcal{A}$ be the collection of subsets of $\mathbb{N}$ which have an asymptotic density.

(a) Give an example of a subset of $\mathbb{N}$ which does not belong to $\mathcal{A}$.

(b) Show that $\mathcal{A}$ is not a σ-algebra.