Stochastic Simulation

Autumn Semester 2024

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Project - 9

Submission deadline: 16 January 2025

Monte Carlo estimation of Sensitivities in Finance

This project aims to explore the applicability of Monte Carlo methods for estimating "sensitivities" as quantities that appear in several tasks in financial computations. Given a financial model, we mean by sensitivities the derivatives of quantities of interest with respect to parameteres such as volatility, stock price, interest rate, etc. Roughly speaking, one is typically interested in quantifying how sensitive a quantity of interest (QoI), e.g. the price of a call option, is with respect to those parameters. These QoI are commonly expressed as expectations of output quantities of stochastic differential equations and therefore, Monte Carlo estimation is a natural approach to compute them.

Note: *This project requires several pen and paper computations. Since the maximum length of the report is set in the project rules, please avoid reporting all of them in detail, and rather focus your report on commenting the numerical results obtained.*

1 Asset price model

The standard and perhaps the most common SDE that appears in the famous Black-Scholes model is a geometric Brownian motion, given by

$$dS_t = rS_t dt + \sigma S_t dW_t, \quad t > 0, \quad S_0 \text{ given} \tag{1}$$

where r is the interest rate, σ is the volatility, both assumed here to be constant, and W_t is a standard Brownian motion. Using Itô's formula we can derive the SDE for $X_t = \log(S_t/S_0)$, that is

$$dX_t = (r - \frac{1}{2}\sigma^2)dt + \sigma dW_t, \quad t > 0, \; X_0 = 0 \tag{2}$$

for which the solution at time $t = T$ can be easily obtained by integration and is

$$X_T = (r - \frac{1}{2}\sigma^2)T + \sigma W_T \tag{3}$$

therefore

$$S_T = S_0 \exp\left\{(r - \frac{1}{2}\sigma^2)T + \sigma W_T\right\}. \tag{4}$$

We are normally interested in the expected value of some function of S_T, say

$$I = \mathbb{E}[f(S_T)] = \int_{\mathbb{R}} f\left(S_0 \exp\left\{(r - \sigma^2/2)T + \sigma w\right\}\right) p_{W_t}(w)dw \tag{5}$$

where $p_{W_t}(\cdot)$ is the probability density function of a $\mathcal{N}(0,T)$ Gaussian random variable. Equivalently, by applying a change of variables, we can express the integral as an expectation with respect to the probability density of S_T, obtaining

$$I = \mathbb{E}[f(S_T)] = \int_0^\infty f(s)\, p_{S_T}(s) ds, \tag{6}$$

where, setting $w = \left(\log(s/S_0) - (r - \sigma^2/2)T\right)/\sigma$, we have

$$p_{S_T}(s) = p_{W_T}(w) \left(\frac{\partial S_T}{\partial W_T}\right)^{-1}(s) = \frac{1}{s\sigma\sqrt{2\pi T}} \exp\left[-\frac{1}{2}\left(\frac{\log(s/S_0) - (r - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right)^2\right]. \tag{7}$$

2 Estimating sensitivities

From the above, it can be understood that the constant quantities S_0, σ, r are parameters that can generally affect the outcome of S_t and therefore $\mathbb{E}[f(S_T)]$. The degree of variability of $I = \mathbb{E}[f(S_T)]$ when those parameters change can be measured by the derivative of I with respect to the corresponding parameters, therefore we call these derivatives as *sensitivities* as they provide us with a measure of how sensitive the quantity of interest is. In the following, we present three main techniques to estimate the sensitivities. More details on Monte Carlo strategies applied on each method can be found in [1, Chapter 7].

2.1 Finite differences

The most straightforward approach to estimate the derivatives of I with respect to some parameter θ is to apply finite differences, that is to compute

$$\frac{\partial I}{\partial \theta} \approx \frac{I(\theta + \Delta\theta) - I(\theta - \Delta\theta)}{2\Delta\theta} \tag{8}$$

and

$$\frac{\partial^2 I}{\partial \theta^2} \approx \frac{I(\theta + \Delta\theta) - 2I(\theta) + I(\theta - \Delta\theta)}{(\Delta\theta)^2}. \tag{9}$$

In the context of Monte Carlo estimation, one can estimate all of the terms $I(\theta+\Delta\theta), I(\theta), I(\theta-\Delta\theta)$ using the same Monte Carlo sample of W_T and the respective S_T.

2.2 Pathwise derivatives

Assuming that the derivative and the integral are interchangeable (which is not always true), that is

$$\frac{\partial}{\partial \theta}\mathbb{E}\left[f(S_T)\right] = \mathbb{E}\left[\frac{\partial}{\partial \theta} f(S_T)\right], \tag{10}$$

we can compute the first derivative of I as

$$\frac{\partial I}{\partial \theta} = \mathbb{E}\left[f'(S_T)\frac{\partial S_T}{\partial \theta}\right] \tag{11}$$

and the second derivative as

$$\frac{\partial^2 I}{\partial \theta^2} = \mathbb{E}\left[f''(S_T)\frac{\partial S_T}{\partial \theta} + f'(S_T)\left(\frac{\partial S_T}{\partial \theta}\right)^2 \right] \tag{12}$$

where the derivative of the stochastic process S_t with respect to θ, called the pathwise derivative, corresponds to $\lim_{h\to 0}(S_t(\theta+h) - S_t(\theta))/h$, if the limit exists almost surely. In this case, one can write a corresponding SDE for such derivatives. Again, in a Monte Carlo setting one will approximate the expectations in (11) and (12) by sample averages.

2.3 Likelihood ratio

In the case where the change of variable has been applied and the interchange of the derivative and the integral is again possible, we can write

$$\frac{\partial I}{\partial \theta} = \int_0^\infty f(s)\frac{\partial p_{S_T}(s)}{\partial \theta} ds = \mathbb{E}\left[f(S_T)\frac{\dot{p}_\theta(S_T)}{p_{S_T}(S_T)} \right], \tag{13}$$

where $\dot{p}_\theta(s) = \frac{\partial p_{S_T}(s)}{\partial \theta}$. The quantity $\frac{\dot{p}_\theta(s)}{p_{S_T}(s)} = \frac{\partial}{\partial \theta}\log p_{S_T}(s)$ is also referred to as *score function*.

3 Application to option pricing and goals of the miniproject

3.1 European call option

Consider the quantity

$$f(S_T) = e^{-rT}\left[S_T - K\right]^+, \tag{14}$$

where $[x]^+ = \max\{x, 0\}$, interest rate $r = 0.05$, maturity time $T = 1$, volatility $\sigma = 0.25$ initial asset price $S_0 = 100$ and strike price $K = 120$.

We are interested in estimating the following quantities shown in Table 1, known as "**the Greeks**" in finance:

Name	Symbol	Definition
Delta	δ	$\partial f(S_T)/\partial S_0$
Vega	ν	$\partial f(S_T)/\partial \sigma$
Gamma	γ	$\partial^2 f(S_T)/\partial S_0^2$

Table 1: Greeks in finance. See [2] for a complete list.

(a) Estimate δ and ν using the finite difference, pathwise derivative and likelihood ratio (LR) methods and plot the estimates together with a confidence interval as a function of the sample size $N \in [10^3, 10^6]$. For the finite difference method, consider different values of h and comment on the bias as h decreases. In this case, would an estimator based on two independent iid samples to estimate $I(\theta + \Delta\theta)$ and $I(\theta - \Delta\theta)$ perform better or worse? Give some theoretical arguments to answer this question.

(b) Estimate γ by applying all possible combinations of pathwise derivative and LR methods in the computation of the first and second derivatives or comment on the one(s) that are not applicable. Plot the estimates and an error bound as a function of N, as above. Comment on the results.

3.2 Mixed estimators for Digital call option

Besides estimating Greeks that involve second derivatives of an asset price model (see point (b) above), another case where different methods can be combined together is when a discontinuity is present. Consider the price of the Digital call option, defined as

$$f(S_T) = e^{-rT}\mathbb{1}_{\{S_T>K\}}. \tag{15}$$

In order to get a δ estimator on the above, one can write

$$\mathbb{1}_{\{x>K\}} = f_\epsilon(x) + (\mathbb{1}_{\{x>K\}} - f_\epsilon(x)) = f_\epsilon(x) + h_\epsilon(x), \tag{16}$$

where $f_\epsilon(x)$ is a continuous approximation of $\mathbb{1}_{\{x>K\}}$, e.g., $f_\epsilon(x) = \min\{1, \max\{0, \frac{x-K+\epsilon}{2\epsilon}\}\}$ and then we can apply pathwise differentiation for $\frac{\partial}{\partial S_0}\mathbb{E}[f_\epsilon(S_T)]$ and likelihood ratio for $\frac{\partial}{\partial S_0}\mathbb{E}[h_\epsilon(S_T)]$

(c) Estimate δ using the above estimator for $\epsilon = 20$, $S_0 = K = 100$, $T = 0.25$ and the remaining parameters as before. Use a pilot run to find how to split optimally a sample of size N between the two parts of the estimator.

(d) Plot the variance of the estimator (with optimal sample splitting) as a function of $\epsilon \in [0, 80]$ and find ϵ (approximately) that achieves the minimum (the case $\epsilon = 0$ corresponds to using only the LR method on $f(S_T)$).

3.3 Path-dependent option

Consider now an Asian option having payoff

$$f(S_{T/m}, S_{2T/m}, \dots, S_T) = e^{-rT}[\bar{S}_T - K]^+ \tag{17}$$

where $\bar{S}_T = \frac{1}{m}\sum_{i=1}^m S_{iT/m}$ is the discrete monitoring average in the time interval [0,T]. It is clear that the price is now dependent on the path of S_t and not only on the final asset price S_T.

(e) Describe how the pathwise derivative and likelihood ratio methods could be applied in this case to compute the Delta and Vega Greeks.

References

[1] S. Asmussen and P.W. Glynn. *Stochastic simulation: algorithms and analysis (Vol. 57).* Springer Science & Business Media, 2007.

[2] Erik Banks and Paul Siegel. *The options applications handbook: hedging and speculating techniques for professional investors.* McGraw-Hill Professional, 2006.