MATH-414 – Stochastic simulation

Lecture 5: Variance Reduction Techniques I

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Outline

Variance Reduction Techniques

Antithetic variables

(Crude) Monte Carlo estimator

Standard setting:

- Z: output of a stochastic model, with $\sigma^2 := \mathbb{V}\mathrm{ar}[Z] < \infty$; exact distribution not known, but Z can be simulated exactly.
- Typically, $Z = \phi(U_1, U_2, \dots, U_d)$ with $(U_1, \dots, U_d) \overset{\text{iid}}{\sim} \mathcal{U}(0,1)$
- **Goal**: estimate $\mu = \mathbb{E}[Z] = \int_{[0,1]^d} \phi(u_1, \dots, u_N) du_1 \dots du_d$

Crude Monte Carlo (CMC) estimator:

$$\hat{\mu}_N = \frac{1}{N} \sum_{i=1}^{N} Z^{(i)}, \quad \text{with} \quad Z^{(1)}, \dots, Z^{(N)} \overset{\text{iid}}{\sim} Z$$

- **Mean Squared Error**:

$$\mathrm{MSE}(\hat{\mu}_N) = \mathbb{E}\left[(\hat{\mu}_N - \mu)^2\right] = \mathbb{V}\mathrm{ar}[\hat{\mu}_N] = \frac{\mathbb{V}\mathrm{ar}[Z]}{N}$$

- **Asymptotic confidence interval**:

$$|\hat{\mu}_N - \mu| \leq c_{1-\frac{\alpha}{2}} \frac{\sqrt{\mathbb{V}\mathrm{ar}[Z]}}{\sqrt{N}} \quad \text{with prob. } 1-\alpha \text{ as } N \to \infty$$

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Variance Reduction Techniques

The error of the CMC estimator is proportional to $\sqrt{\mathbb{V}\mathrm{ar}\,[Z]}$.

Variance Reduction Techniques aim at reducing this multiplicative constant. **How can this be done**?

Idea:

- Find another random variable $\tilde{Z}$ such that

$$\mathbb{E}[\tilde{Z}] = \mathbb{E}\,[Z] = \mu \quad \text{and} \quad \mathbb{V}\mathrm{ar}[\tilde{Z}] \ll \mathbb{V}\mathrm{ar}\,[Z]$$

- Then apply CMC on $\tilde{Z}$:

$$\hat{\mu}_{VR} = \frac{1}{N}\sum_{i=1}^{N} \tilde{Z}^{(i)}, \quad \text{with} \quad \tilde{Z}^{(1)}, \dots, \tilde{Z}^{(N)} \overset{\text{iid}}{\sim} \tilde{Z}$$

so that $$\mathrm{MSE}(\hat{\mu}_{VR}) = \frac{\mathbb{V}\mathrm{ar}[\tilde{Z}]}{N} \ll \frac{\mathbb{V}\mathrm{ar}[Z]}{N}$$

Variance reduction techniques act on the multiplicative constant, not on the MC rate $\frac{1}{\sqrt{N}}$!

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Antithetic variables

We call Z_a an antithetic variable of Z if

- $Z_a \sim Z$ (i.e. Z and Z_a have the same distribution
- $\mathrm{Cov}(Z, Z_a) < 0$ (Z and Z_a are negatively correlated)

Then we can build the following estimator (assuming N even)

- Generate $N/2$ iid pairs $(Z^{(i)}, Z_a^{(i)}) \sim (Z, Z_a)$, $i = 1, \ldots, N/2$
- Define $\hat{Z}^{(i)} = \frac{Z^{(i)} + Z_a^{(i)}}{2}$
- $\hat{\mu}_{AV} = \frac{1}{N/2} \sum_{i=1}^{N/2} \hat{Z}^{(i)} = \frac{1}{N} \sum_{i=1}^{N/2} (Z^{(i)} + Z_a^{(i)})$

Variance of the estimator:

$$
\begin{aligned}
\mathbb{V}\mathrm{ar}\left[\hat{\mu}_{AV}\right] &= \frac{1}{N^2} \sum_{i=1}^{N/2} \mathbb{V}\mathrm{ar}\left[Z^{(i)} + Z_a^{(i)}\right] = \frac{1}{2N} \mathbb{V}\mathrm{ar}\left[Z + Z_a\right] \\
&= \frac{1}{2N} \left(\mathbb{V}\mathrm{ar}\left[Z\right] + \mathbb{V}\mathrm{ar}\left[Z_a\right] + 2\,\mathrm{Cov}(Z, Z_a)\right) \\
&= \frac{\mathbb{V}\mathrm{ar}\left[Z\right] + \mathrm{Cov}(Z, Z_a)}{N} < \mathbb{V}\mathrm{ar}\left[\hat{\mu}_{\mathrm{CMC}}\right]
\end{aligned}
$$

Antithetic variables

How do we construct in practice Z_a?

Suppose that $Z = \phi(U_1, \dots, U_d)$ with $U_1, \dots, U_d \overset{\text{iid}}{\sim} \mathcal{U}([0,1])$. Then

$$Z_a := \phi(1 - U_1, \dots, 1 - U_d) \sim Z$$

More generally, suppose that $Z = \phi(X_1, \dots, X_d)$ with

- $X_1, \dots, X_d$ independent random variables
- $2\mathbb{E}[X_i] - X_i \sim X_i$ (i.e. X_i's law is symmetric around the mean)

Then

$$Z_a = \phi(2\mathbb{E}[X_1] - X_1, \dots, 2\mathbb{E}[X_d] - X_d) \sim Z$$

When is $\mathrm{Cov}(Z, Z_a) \leq 0$?

Proposition. (Sufficient condition) *If*

- $X_1, \dots, X_d$ *are independent r.vs and* $2\mathbb{E}[X_i] - X_i \sim X_i, \ \forall i$
- ϕ *is a monotonic in each of its arguments*

Then $Z = \phi(X_1, \dots, X_d)$ *and* $Z_a = \phi(2\mathbb{E}[X_1] - X_1, \dots, 2\mathbb{E}[X_d] - X_d)$ *satisfy*

$$\mathbb{E}[Z] = \mathbb{E}[Z_a] \quad \text{and} \quad \mathrm{Cov}(Z, Z_a) \leq 0.$$

Proof in 1D

Let f be the pdf of X and $\psi(X) = \phi(2\mathbb{E}[X] - X)$. Observe that if ϕ is non-decresing, then ψ is non-increasing and vice-versa.

Denote, moreover, $\tilde{\phi}(X) = \phi(X) - \mathbb{E}[\phi(X)]$ and similarly for $\tilde{\psi}$

$$
\begin{aligned}
\mathrm{Cov}(Z, Z_a) = \mathbb{E}\left[\tilde{\phi}(X)\tilde{\psi}(X)\right] &= \int \tilde{\phi}(x)\tilde{\psi}(x) f(x) dx \\
&= \frac{1}{2} \iint (\tilde{\phi}(x) - \tilde{\phi}(y))(\tilde{\psi}(x) - \tilde{\psi}(y)) f(x) f(y) dxdy \\
&= \frac{1}{2} \iint (\phi(x) - \phi(y))(\psi(x) - \psi(y)) f(x) f(y) dxdy
\end{aligned}
$$

Assume ϕ non-decreasing (hence ψ non-increasing). Then

$$
\begin{aligned}
\mathrm{Cov}(Z, Z_a) =& \frac{1}{2} \iint_{x \geq y} \underbrace{(\phi(x) - \phi(y))}_{\geq 0} \underbrace{(\psi(x) - \psi(y))}_{\leq 0} f(x) f(y) dxdy \\
&+ \frac{1}{2} \iint_{x < y} \underbrace{(\phi(x) - \phi(y))}_{\leq 0} \underbrace{(\psi(x) - \psi(y))}_{\geq 0} f(x) f(y) dxdy \leq 0
\end{aligned}
$$

Antithetic variables – algorithm

Algorithm: Antithetic variables.

1 Generate $N/2$ iid replicas $X^{(1)}, \dots, X^{(N/2)}$ of X;

2 For each $X^{(i)}$ compute $Z^{(i)} = \psi(X^{(i)})$ and $Z_a^{(i)} = \psi(2\mathbb{E}[X] - X^{(i)})$;

3 Compute $\hat{\mu}_{\mathrm{AV}} = \frac{1}{N}\sum_{i=1}^{N/2}(Z^{(i)} + Z_a^{(i)})$.

4 Estimate $\hat{\sigma}_{\mathrm{AV}}^2 = \frac{1}{N/2-1}\sum_{i=1}^{N/2}\left(\frac{Z^{(i)}+Z_a^{(i)}}{2} - \hat{\mu}_{\mathrm{AV}}\right)^2$

5 Output $\hat{\mu}_{\mathrm{AV}}$ and a (asymptotic) $1-\alpha$ confidence interval

$$\hat{I}_{\alpha,N} = \left[\hat{\mu}_{\mathrm{AV}} - c_{1-\alpha/2}\frac{\hat{\sigma}_{\mathrm{AV}}}{\sqrt{N/2}},\ \ \hat{\mu}_{\mathrm{AV}} + c_{1-\alpha/2}\frac{\hat{\sigma}_{\mathrm{AV}}}{\sqrt{N/2}}\right]$$

Example 1 – option pricing

- S_t: value of an asset at time t, modeled by

$$dS_t = rS_t\,dt + \sigma S_t\,dW_t, \quad t \in (0,T]$$

- $\psi : \mathbb{R} \to \mathbb{R}$: option's payoff (increasing function)
- **Goal**: estimate option price $\mu = \mathbb{E}\left[e^{-rT}\psi(S_T)\right]$

The log-price $X_t = \log(S_t/S_0)$ satisfies

$$dX_t = (r - \sigma^2/2)\,dt + \sigma\,dW_t, \quad X_0 = 0.$$

Hence

$$X_T = (r - \sigma^2/2)T + \sigma W_T \sim N((r - \sigma^2/2)T, \sigma^2 T)$$

Defining $Z = e^{-rT}\psi(S_T) = e^{-rT}\psi(S_0 e^{X_T}) = \tilde{\psi}(X_T)$ we have

- X_T has a normal distribution (symmetric around the mean)
- $\tilde{\psi}$ is increasing (composition of increasing functions)

The antithetic variables estimator

$$\hat{\mu}_{\mathrm{AV}} = \frac{1}{N}\sum_{i=1}^{N/2}\left(\tilde{\psi}(X_T^{(i)}) + \tilde{\psi}((2r - \sigma^2)T - X_T^{(i)})\right), \qquad X_T^{(i)} \overset{\text{iid}}{\sim} X_T$$

will therefore lead to variance reduction.

Example 2 – random walk

Consider a random walk on the integers:

$$Y_{n+1} = Y_n + X_{n+1}, \quad Y_0 = 0$$

with X_i iid, $\mathbb{P}(X_i = 1) = \mathbb{P}(X_i = -1) = 1/2$
Goal: for some $s \in \mathbb{N}$, estimate

$$\mu = \mathbb{P}(Y_N \geq s) = \mathbb{E}[Z], \quad \text{with} \quad Z = \mathbb{1}_{\{Y_N \geq s\}}$$

How to construct an antithetic variable estimator?

We have $\quad Z = \mathbb{1}_{\{Y_N \geq s\}} = \mathbb{1}_{\{\sum_{n=1}^N X_n \geq s\}} = \psi(X_1, \ldots, X_N)$

Moreover,

- X_i is symmetric around its mean $\mathbb{E}[X_i = 0]$
- ψ is non-decreasing in each argument

Hence, $Z_a = \psi(-X_1, \ldots, -X_N)$ is negatively correlated with Z

MC estimator with antithetic variables:

- generate $N/2$ iid paths $Y_{n+1}^{(i)} = Y_n^{(i)} + X_n^{(i)}$ and corresponding antithetic paths $\tilde{Y}_{n+1}^{(i)} = \tilde{Y}_n^{(i)} - X_n^{(i)}$,
- build estimator $\hat{\mu}_{\text{AV}} = \frac{1}{N} \sum_{i=1}^{N/2} (\mathbb{1}_{\{Y_N^{(i)} \geq s\}} + \mathbb{1}_{\{\tilde{Y}_N^{(i)} \geq s\}})$.

Exercise: what if $\mathbb{P}(X_i = 1) = a$, $\mathbb{P}(X_i = -1) = 1 - a$ with $a \neq \frac{1}{2}$?