Stochastic Simulation

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Derivative estimation

Exercise 1

Consider a simple PERT network problem where

$$z(\theta) = \mathbb{E}\left[\max\{\theta X_1 + X_2, (1-\theta)X_3\}\right],$$

with $0 < \theta < 1$. The random variables X_1, X_2, and X_3 follow Erlang(2) distributions with the following properties with means 1,2, and 3, respectively. We consider the optimization problem

$$\min_{\theta \in [0,1]} z(\theta). \tag{1}$$

1. Verify that $z'(\theta) = \mathbb{E}[\frac{d}{d\theta} \max\{\theta X_1 + X_2, (1-\theta)X_3\}]$. Write pen and paper the estimators IPA and LR to approximate $z'(\theta)$.

2. Estimate $z'(\theta)$ using IPA and LR with 10^5 samples and for a grid $\theta \in \{0.1, 0.2, \ldots, 0.9\}$. For each value of θ estimate the standard deviation of the estimators of $z'(\theta)$ and plot them as a function of θ.

3. Implement the stochastic gradient descent method to minimize $z(\theta)$. Use a step size of your choice and compute the approximated optimal $\hat{\theta}^*$. Use both
   - SGD with a decreasing step size $\tau_k \propto \frac{1}{k}$ and a fixed sample size to estimate z' at each iteration;
   - SGD with a fixed step size τ and a geometrically increasing sample size to estimate z' at each iteration.
   - Using 0.6253 as a reference solution, plot the error $|\theta_k - \theta^*|$ as a function of the number of gradient evaluations.

Exercise 2

Consider the problem of estimating

$$\frac{d}{d\theta} I(\theta) \tag{2}$$

where

$$I(\theta) = \mathbb{E}[\mathbb{1}_{\{\theta X > 1\}}] \tag{3}$$

and $X \sim \mathcal{N}(0,1)$. Since the indicator function is discontinuous, we may consider the smoothed version of the integral defined by

$$I_\varepsilon(\theta) = \mathbb{E}\left[\Phi\left(\frac{(\theta X - 1)}{\varepsilon}\right)\right], \tag{4}$$

where ϕ denotes the CDF of a standard normal random variable. Address the following points.

1. Compute the analytic value of $\frac{d}{d\theta}I(\theta)$ and $\frac{d}{d\theta}I_\varepsilon(\theta)$.
2. Implement the IPA method to compute an approximation of $I_\varepsilon(\theta)$. Using the previously computed values, assess the Monte Carlo error and the error with respect to the smoothing parameter ε.
3. Implement the LR method on the non-regularized function $I(\theta)$.