

Stochastic Simulations

Autumn Semester 2024

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Lab 09 – 14 November 2024

Markov Chains

We recall some concepts on the theory of Markov chains on a discrete state space $\mathcal{X}$.

Irreducibility

Let P be a transition matrix on $\mathcal{X}$. We say that a state $x_i \in \mathcal{X}$ communicates with a state $x_j \in \mathcal{X}$ if $\mathbb{P}(X_n = x_j, \text{ for some } n \mid X_0 = x_i) > 0$; equivalently, if $\exists n \geq 0$ such that $P_{i,j}^{(n)} > 0$. A Markov chain is *irreducible* if every state x_j communicates with every other state x_i , i.e.,

$$\forall i, j, \quad \exists n \geq 0 \text{ such that } P_{i,j}^{(n)} > 0.$$

Recurrence

A state $x_i \in \mathcal{X}$ is *recurrent* if $\mathbb{P}(X_n = x_i \text{ infinitely often}) = 1$, that is, x_i is visited infinitely-often with probability 1. A Markov chain $\{X_n\}$ is recurrent if every state is recurrent. It is known that every irreducible recurrent Markov chain $\{X_n\}$ on a discrete state space has a (not necessarily finite) invariant distribution π that is unique up to a multiplicative constant. However, if the state space is *finite*, every irreducible Markov chain $\{X_n\}$ is recurrent and has a unique invariant probability distribution.

Aperiodicity

The *period* of a state x_i is the largest integer d satisfying the following property: $P_{i,i}^{(n)} = 0$, whenever n is not divisible by d . The period of x_i is given by $d(i)$. We say that if $d(i) > 1$, then the state x_i is *periodic*. We say that the state x_i is *aperiodic* otherwise. If a Markov chain $\{X_n\}$ is irreducible and has an aperiodic state, then all states are aperiodic, in which case we say that $\{X_n\}$ is aperiodic. In particular, an irreducible Markov chain $\{X_n\}$ is aperiodic if there exists a state $x_j \in \mathcal{X}$ such that $P_{jj} > 0$. It is known that an irreducible Markov chain $\{X_n\}$ on a *finite* state space $\mathcal{X}$ converges to π , i.e., $\pi_j = \lim_{n \rightarrow \infty} \mathbb{P}(X_n = x_j)$, $x_j \in \mathcal{X}$, if and only if $\{X_n\}$ is aperiodic.

Exercise 1

Consider the random walk $\{X_n \in \mathbb{Z}, n \in \mathbb{N}_0\}$ with $X_0 \sim \lambda$ on the lattice $\mathcal{X} := \{i: i \in \mathbb{Z}, |i| \leq 2N^2\}$, whose transition probabilities are given by

$$\begin{aligned}\mathbb{P}(X_{n+1} = i \pm 1 | X_n = i) &= \alpha \left(1 \mp \frac{i}{2N^2} \right), \quad |i| \leq 2N^2, \\ \mathbb{P}(X_{n+1} = i | X_n = i) &= 1 - 2\alpha,\end{aligned}$$

for some $\alpha \in]0, \frac{1}{2}]$ and $N \in \mathbb{N}$.

1. Implement an algorithm that simulates the Markov Chain $\{X_n \in \mathbb{Z}, n \in \mathbb{N}_0\}$. Use your implementation to address the following points for different values of $N \geq 1$:
 - (a) Assess numerically that the Markov chain converges to an invariant distribution by simulating multiple (independent) chains, each starting in 0 (i.e. $\lambda = \delta_0$). That is, monitor the following quantities (rather, suitable Monte Carlo approximations) as functions of the Markov chain length n .
 - i. $\mathbb{E}_\lambda(X_n^p)^{1/p}$ for $p \in \{1, 2, 4\}$,
 - ii. $M_{X_n}(t) := \mathbb{E}_\lambda(e^{tX_n})$ for $t \in [-1, 1]$.

Speculate on the invariant distribution.

- (b) For $N = 10$, compute the eigenvalues and eigenvectors of the transition matrix P . Use the obtained results to deduce the invariant distribution π . **Hint:** Use `np.linalg.eig(P)`.
- (c) Assess the validity of the ergodic theorem. That is, verify that

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^n f(X_n) = \mathbb{E}_\pi(f), \quad \mathbb{P}_\lambda\text{-a.s.},$$

for any $f: \mathcal{X} \rightarrow \mathbb{R}$, with $\sum_n |f(X_n)| \pi_n < \infty$. Specifically, investigate this identity for the moments used in Point 1(a)i and monitor the rate of convergence as a function of n .

2. Consider the rescaled Markov chain $Y_n := \frac{1}{N}X_n$ with state space $\mathcal{Y} := \{x_i \equiv \frac{i}{N}: i \in \mathbb{Z}, |i| \leq 2N^2\}$. Show by means of numerical simulations that the invariant distribution $\nu \equiv \nu_N$ of $\{Y_n \in \mathbb{Z}, n \in \mathbb{N}_0\}$ is an accurate approximation to the standard normal measure. Moreover, illustrate that the approximation quality improves as N increases.

Exercise 2

A random walk on the integers $I = \{0, 1, 2, \dots\}$ can be constructed in the following way. For $0 < p < 1/2$, let $Y_0, Y_1, \dots$ be i.i.d random variables with $P(Y_i = 1) = p$ and $P(Y_i = -1) = 1 - p$. Define two random walks as (1) $X_n = \max\{X_{n-1} + Y_n, 0\}$ and (2) $Z_n = |Z_{n-1} + Y_n|$.

1. Show that (X_n) and (Z_n) are Markov chains.

2. Show that an invariant probability measure of the chains (X_n) and (Z_n) is given by

$$\hat{\pi} = \left[1, \left(\frac{p}{1-p} \right), \dots, \left(\frac{p}{1-p} \right)^k, \dots \right] a_0, \quad k \geq 0$$

$$\bar{\pi} = \left[1, \frac{1}{1-p}, \frac{p}{(1-p)^2}, \dots, \frac{p^{k-1}}{(1-p)^k}, \dots \right] b_0, \quad k \geq 1,$$

respectively. Find p, a_0, b_0 such that the expressions above are probability distributions.

3. Let $p = 1/8$. Assess numerically the convergence of both Markov chains to their invariant distribution by simulating multiple (independent) chains of length $n = 100$, each starting in 0 (i.e. $\lambda = \delta_0$). That is, plot the empirical distribution of X_n, Z_n vs $\hat{\pi}$ and $\bar{\pi}$, respectively. Repeat your experiments for $m = n + 1$. Explain your results.
4. Discuss the periodicity of both chains.

Exercise 3

Given the transition matrix

$$P = \begin{pmatrix} 0.0 & 0.4 & 0.6 & 0.0 & 0.0 \\ 0.65 & 0.0 & 0.35 & 0.0 & 0.0 \\ 0.32 & 0.68 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.12 & 0.88 \\ 0.0 & 0.0 & 0.0 & 0.56 & 0.44 \end{pmatrix}, \quad (1)$$

and examine whether the corresponding chain is irreducible and aperiodic.