Stochastic Simulations

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The Monte Carlo method and Variance Reduction Techniques

Exercise 1

A simulator would like to produce an unbiased estimate of $\mathbb{E}(XY)$, where the two independent random variables X and Y have bounded first moments and can be generated by a stochastic simulation. To this end, she simulates $R \in \mathbb{N}$ replications $X_1, \dots, X_R$ of X and, independently of this, R replications $Y_1, \dots, Y_R$ of Y. She thus has the following two natural estimators for $\mathbb{E}(XY)$ at her disposal:

$$\mathrm{Est}_1 := \left(\frac{1}{R}\sum_{r=1}^{R} X_r\right)\left(\frac{1}{R}\sum_{r=1}^{R} Y_r\right) \quad \text{and} \quad \mathrm{Est}_2 := \frac{1}{R}\sum_{r=1}^{R} X_r Y_r \,.$$

1. Verify that both estimators Est_1 and Est_2 are unbiased.
2. Show that $\mathrm{Var}(\mathrm{Est}_1) < \mathrm{Var}(\mathrm{Est}_2)$.
3. Use the delta method to show that $\sqrt{R}(\mathrm{Est}_1 - \mu_x\mu_y) \xrightarrow{d} N(0, \tau^2)$. Find τ^2 explicitly and derive a $1-\alpha$ asymptotic confidence interval.

Exercise 2

Algorithm 1 proposes a sequential Monte Carlo method to compute the expectation $\mathbb{E}[X]$ of a random variable X, where the sample size is doubled at each iteration until the estimated $1-\alpha$ confidence interval based on a central limit theorem approximation is smaller than a prescribed tolerance ϵ. The algorithm then outputs the final sample size $N(\epsilon, \alpha)$, as well as the estimated value $\bar{X}_N$.

Algorithm 1 can be particularly sensitive to the choice of initial sample size N_0, and as such, we would like to assess the robustness of such an algorithm in estimating $\mathbb{E}[X]$ for different distributions of X. For some values of N_0 ranging between 10 and 50, consider $\alpha = 10^{-1.5}$ and $\epsilon = 1/10$, and the following random variables:

1. $X \sim \mathrm{Pareto}(x_m = 1, \gamma = 3.1)$ (i.e. with PDF $p(y) = \mathbb{1}_{y>x_m} x_m^{\gamma} \gamma y^{-(\gamma+1)}$), $\mathbb{E}[X] = \frac{\gamma x_m}{\gamma - 1}$.
2. $X \sim \mathrm{Lognormal}(\mu = 0, \sigma = 1)$, $\mathbb{E}[X] = \exp\left(\mu + \frac{\sigma^2}{2}\right)$.
3. $X \sim U([-1, 1])$, $\mathbb{E}[X] = 0$.

Algorithm 1 Sample Variance Based SMC

Input: N_0, distribution λ, accuracy $\epsilon > 0$, confidence $1-\alpha > 0$.
Output: $\overline{X}_{\epsilon,\alpha}$ (i.e, approximation of $\mathbb{E}[X]$ with $X \sim \lambda$), N.

Set $k = 0$, generate N_k i.i.d. replica $\{X_i\}_{i=1}^{N_k}$ of $X \sim \lambda$ and

$$\bar{X}_{N_k} = \frac{1}{N_k} \sum_{i=1}^{N_k} X_i, \tag{1}$$

$$\overline{\sigma}^2_{N_k} := \frac{1}{N_k - 1} \sum_{i=1}^{N_k} (X_i - \overline{X}_{N_k})^2. \tag{2}$$

while $\bar{\sigma}_{N_k} C_{1-\alpha/2}/\sqrt{N_k} > \epsilon$ **do**
  Set $k = k+1$ and $N_k = 2N_{k-1}$.
  Generate a new batch of N_k i.i.d. replicas $\{X_i\}_{i=1}^{N_k}$ of $X \sim \lambda$.
  Compute the sample variance $\overline{\sigma}^2_{N_k}$ by (2).
end while
Set $N = N_k$, generate i.i.d. samples $\{X_i\}_{i=1}^{N}$ of λ and compute the output sample mean $\overline{X}_{\epsilon,\alpha}$.

Repeat the simulation $K = 20\alpha^{-1}$ times and record the sample sizes $\{N^{(i)}\}_{i=1}^{K}$ as well as the computed sample means $\{\bar{X}^{(i)}_{\epsilon,\alpha}\}_{i=1}^{K}$ returned by the algorithm for each run $i = 1, ..., K$. Estimate the probability of failure $\overline{p}$ of the algorithm :

$$\overline{p}_K(N_0, \epsilon, \alpha) = \frac{1}{K} \sum_{i=1}^{K} \mathbb{1}_{|\bar{X}^{(i)}_{\epsilon,\alpha} - \mathbb{E}[X]| > \epsilon}.$$

Then check whether $\overline{p}_K(N_0, \epsilon, \alpha) \leq \alpha$ holds. Repeat your experiment for different values of ϵ and α. Discuss your results. **Hint:** You may generate Pareto(x_m, α) r.v. by inversion.

Then, compare Algorithm 1 with the sequential Monte Carlo method in Algorithm 2, where one realization is added at a time.

Exercise 3

Consider the problem of pricing a Barrier option with maturity $T > 0$ based on the stock price S, which is given as the solution to the stochastic differential equation

$$dS = rS\,dt + \sigma S\,dW \ , \quad S(0) = S_0 \ ,$$

where W denotes a standard one-dimensional Wiener process. One can show that $S_t = S_0 e^{X_t}$, where $X_t = (r - \sigma^2/2)t + \sigma W_t$ with W being a standard Wiener process. It follows that S_t has a log-normal distribution for any $t > 0$. For $m \in \mathbb{N}$, let $t_i = i\Delta t$ with $\Delta t = T/m$ denote the discrete observation times of the stock price S (e.g. daily at market closure). The payoff of a call option subject to a lower barrier is then given by

$$\Psi(S_{t_0}, S_{t_1}, \ldots, S_T) = (S_T - K)_+ \mathbb{I}_{\left\{B \leq \min_{i=0,\ldots,m}(S_{t_i})\right\}} \ ,$$

Algorithm 2 One-at-a-time Sample Variance Based SMC

Input: N_0, distribution λ, accuracy $\epsilon > 0$, confidence $1 - \alpha > 0$.

Output: $\overline{X}_{\epsilon,\alpha}$ (i.e, approximation of $\mathbb{E}[X]$ with $X \sim \lambda$), N.

Set $k = 0$, generate N_k i.i.d. samples $\{X_i\}_{i=1}^{N_k}$ of λ and compute the sample variance

$$\overline{\sigma}^2_{N_k} := \frac{1}{N_k - 1} \sum_{i=1}^{N_k} (X_i - \overline{X}_{N_k})^2. \tag{3}$$

while $\bar{\sigma}_{N_k} C_{1-\alpha/2}/\sqrt{N_k} > \epsilon$ **do**

Set $k = k + 1$ and $N_k = N_{k-1} + 1$.

Generate a new i.i.d. sample $X^{(N_k+1)}$ of λ.

Compute

$$\bar{\mu}_{N_k+1} = \frac{N_k}{N_k + 1} \bar{\mu} + \frac{1}{N_k + 1} X^{(N_k+1)} \tag{4}$$

$$\bar{\sigma}^2_{N_k+1} = \frac{N_k - 1}{N_k} \sigma^2_{N_k} + \frac{1}{N_k + 1} (X^{(N_k+1)} - \bar{\mu}_{N_k})^2 \tag{5}$$

end while

Set $N = N_k$, generate i.i.d. samples $\{X_i\}_{i=1}^{N}$ of λ and compute the output sample mean $\overline{X}_{\epsilon,\alpha}$.

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where $B < S_0$ denotes the Barrier and $K \leq S_0$ the strike price. Here, $z_+ = (|z| + z)/2$ denotes the positive part of z. Estimate the expected payoff $\mathbb{E}\big(\Psi(S_{t_0}, S_{t_1}, \ldots, S_T)\big)$ with antithetic variables, using the process parameters $m = 1000$, $r = 0.5$, $\sigma = 0.3$, $T = 2$, $S_0 = 5$, and $K = 10$. Specifically, investigate the variance reduction effect for different barrier values B.