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Statistical Machine Learning

A Note on B-Splines

A *spline* of degree $d \geq 0$ is a piecewise polynomial function of degree d. The points at which the polynomials meet are known as *knots* and at every knot where two polynomials meet their first $(d-1)$ derivatives are equal.

The set of splines of a given degree d and a given set of knots $\{\xi_j\}_{j=0}^{K+1}$ is a finite-dimensional vector space and hence admits a basis. Thus every spline in this set can be represented as a linear combination of a finite number of splines we call basis splines. Of course, there are many possible choices of basis functions. We shall now construct a particularly nice basis called the *B-spline* basis.

Note that by convention $\{\xi_j\}_{j=0}^{K+1}$ are written in an increasing order. Thus, $\xi_j < \xi_{j+1}$ for $0 \leq j \leq K$. The construction proceeds in the following steps:

1. We define τ_j as follows:

$$\tau_j = \begin{cases} \xi_0 & 0 \leq j \leq d \\ \xi_{j-d} & d+1 \leq j \leq d+K \\ \xi_{K+1} & K+d+1 \leq j \leq K+2d+1 \end{cases}$$

   Notice how we repeat the outer knots ξ_0 and ξ_{K+1} both $(d+1)$ times while the inner knots $\{\xi_j\}_{j=1}^{K}$ have only one of the τ_js equal to them.

2. The B-splines of degree 0 are given by

$$B_{j,0}(x) \coloneqq \mathbf{1}_{\{\tau_j \leq x < \tau_{j+1}\}}(x)$$

   when $\tau_j < \tau_{j+1}$ and zero otherwise. Note that although the "splines" $B_{j,0}$ which are zero thoughout do not actually form part of the basis for splines of degree 0, they aid our construction of higher degree splines.

3. The B-splines of higher degrees are constructed recursively using the Cox-de Boor recursion formula given by

$$B_{j,k+1} \coloneqq \omega_{j,k}(x) B_{j,k}(x) + [1 - \omega_{j+1,k}(x)] B_{j+1,k}(x)$$

   for $0 \leq k < d$ where

$$\omega_{j,k}(x) = \begin{cases} \frac{x-\tau_j}{\tau_{j+k+1}-\tau_j} & \tau_j < \tau_{j+k+1} \\ 0 & \text{otherwise.} \end{cases}$$

To conclude, the B-splines of degree d, given by $\{B_{j,d}\}_{j=0}^{K+1}$ as defined above, form a basis for splines of degree d with the knots $\{\xi_j\}_{j=0}^{K+1}$.