Take-home assignment

You are supposed to hand over solutions by 12:00 of November 21 to Max or Ilaria. You can either hand over written or printed solutions, or send them in the PDF format by e-mail to Max.

Every exercise is assigned a number of points. You need to gain 7 *points* to get the full grade. The choice of exercises is up to you; it is not necessary to do all of them. The result of the take home assignment constitutes $\frac{1}{6}$-th of the final grade.

As in the lectures we denote by Ab the category of abelian groups, by Ab_X the category of sheaves (of abelian groups) on a topological space X, and by PAb_X the category of presheaves of abelian groups on X. For each topolgical space X and each abelian group A we denote by $H^i(X, A)$ the i-th sheaf cohomology of X with coefficients in the constant sheaf $\underline{A}_X$.

The first exercise did not appear in the homework before, and thus carries extra points.

Exercise 1. (6 *points*) For each $n \geqslant 1$ let S^n be the topological n-dimensional sphere, so that S^1 is the disjoint union of 2 points, and S^2 is the topological Riemann sphere. Following the example in the lectures calculate the sheaf cohomology $H^i(S^n, \mathbf{Z})$ for all $n \geqslant 2$ and $i \geqslant 0$:

- Decompose S^n into a suitable union of contractible closed subsets.
- Construct an appropriate short exact sequence of sheaves (and prove that it is short exact!). Use the resulting long exact sequence of cohomology to compute the cohomology groups $H^i(S^n, \mathbf{Z})$ by induction on n.

The case $n = 1$ was done in the lectures. You are free to invoke it without proof.

The remaining exercises have appeared in the homework before.

Exercise 2. (2 *points*) Let A be an abelian group and let X be a topological space. Consider a presheaf $P_{A,X}$ defined by the formula $P_{A,X}(U) := A$. Consider also the *constant sheaf* $\underline{A}_X$ defined by the formula

$$\underline{A}_X(U) := \{ \text{ locally constant maps } f \colon U \to A \ \}.$$

(Recall that a map is *locally constant* if it is continuous w.r.t. the discrete topology on A.) Construct an isomorphism of sheaves

$$P_{A,X}^{\#} \xrightarrow{\sim} \underline{A}_X.$$

Exercise 3. (1 *point*) Let X be the closed unit interval $[0, 1]$. Consider a presheaf $\mathcal{F}$ on X defined by the formula

$$\mathcal{F}(U) := \{ \text{ bounded continuous functions } f \colon U \to \mathbf{C} \ \}.$$

Is this presheaf a sheaf? Prove your answer.

Exercise 4. (2 *points*) Let X be the closed interval $[0, 1]$.

(a) Show that up to isomorphism there is a unique sheaf $\mathcal{F}$ on X with stalks $\mathcal{F}_0 = \mathcal{F}_1 = \mathbf{Z}$ and $\mathcal{F}_x = 0$ for all $x \in (0, 1)$.

(b) For each abelian group A calculate the Hom groups

$$\mathrm{Hom}(\mathcal{F}, \underline{A}_X) \quad \text{and} \quad \mathrm{Hom}(\underline{A}_X, \mathcal{F}).$$

Exercise 5. (2 *points*) Let $\mathcal{A}$ be an abelian category, and let $I^\bullet$ be a complex of injective objects of $\mathcal{A}$. Suppose that the complex $I^\bullet$ is

- *bounded below*, i.e. $I^n = 0$ for all $n \ll 0$, and is
- *exact*, i.e. $H^n(I^\bullet) = 0$ for all n.

Show that in the homotopy category $K(\mathcal{A})$ we have

$$I^\bullet = 0,$$

i.e. the complex $I^\bullet$ is a zero object.

Exercise 6. (2 *points*) Let $n > 1$ be a positive integer, and consider the functor $F\colon Ab \to Ab$ that sends an abelian group A to the subgroup nA.

(a) Show that the functor F transforms monomorphisms to monomorphisms, and epimorphisms to epimorphisms.

(b) Show that the functor F is neither left nor right exact.

Exercise 7. (2 *points*) Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories and let $F\colon \mathcal{A} \to \mathcal{B}$ be an equivalence of categories. We *do not* assume a priori that F is additive.

(a) Show that the functor F is additive.

(b) Show that the functor F is exact, i.e. transforms short exact sequences to short exact sequences. Hint: Deduce separately that F preserves kernels and cokernels.

Exercise 8. (3 *points*) Let X be a topological space. Let $U \subset X$ be an open subset, and let $\underline{\mathbf{Z}}_{X,U}$ be the sheafification of the presheaf

$$P_{X,U}\colon V \longmapsto \begin{cases} \mathbf{Z}, & V \subset U, \\ 0, & V \not\subset U. \end{cases}$$

(a) For each sheaf $\mathcal{F}$ on X construct a natural isomorphism

$$\mathcal{F}(U) \cong \mathrm{Hom}(\underline{\mathbf{Z}}_{X,U}, \mathcal{F}).$$

(b) Let $\mu\colon P_{X,U} \to P_{X,X}$ be the morphism of presheaves such that $\mu_V = 1$ when $V \subset U$ and $\mu_V = 0$ otherwise. Show that the induced morphism of sheaves $\underline{\mathbf{Z}}_{X,U} \to \underline{\mathbf{Z}}_X$ is a monomorphism.

(c) Show that we have a commutative square

$$\begin{tikzcd} \mathrm{Hom}(\underline{\mathbf{Z}}_X, \mathcal{F}) \arrow[r, "(b)"] \arrow[d, "(a)"'] & \mathrm{Hom}(\underline{\mathbf{Z}}_{X,U}, \mathcal{F}) \arrow[d, "(a)"] \\ \mathcal{F}(X) \arrow[r, "\text{restriction}"] & \mathcal{F}(U). \end{tikzcd}$$

(d) Now let $\mathcal{I}$ be an injective sheaf. Deduce that for every open $U \subset X$ the morphism $\mathcal{I}(X) \twoheadrightarrow \mathcal{I}(U)$ is surjective, and concude that $\mathcal{I}$ is flasque.

Exercise 9. (2 *points*) Consider the open subset

$$U := \{x + iy \in \mathbf{C} \mid x, y \in \mathbf{R},\ y > 0\}.$$

Show that U is isomorphic as a Riemann surface to the open unit disk Δ.

Exercise 10. (2 *points* for the case $n = 1$, or 3 *points* for arbitrary $n \geqslant 1$)
Let V be a $\mathbf{C}$-vector space of dimension $n + 1 \geqslant 1$. Recall that we defined $\mathbf{P}(V)$ to be the set of 1-dimensional $\mathbf{C}$-vector subspaces of V equipped with the quotient topology via the surjection

$$V \smallsetminus \{0\} \longrightarrow\!\!\!\!\!\rightarrow \mathbf{P}(V), \quad v \mapsto \mathbf{C}v.$$

Next, for each hyperplane $H \subset V$, i.e. a $\mathbf{C}$-vector subspace of codimension 1, we set

$$\mathbf{P}(V \smallsetminus H) := \{L \in \mathbf{P}(V) \mid L \not\subset H\}.$$

(a) Show that each $\mathbf{P}(V \smallsetminus H)$ is open in $\mathbf{P}(V)$.

(b) Find a set of $n + 1$ hyperplanes H such that $\mathbf{P}(V \smallsetminus H)$ cover $\mathbf{P}(V)$.

Pick a splitting $s\colon V \twoheadrightarrow H$, i.e. a $\mathbf{C}$-linear map such that $s|_H = 1_H$. Pick also an isomorphism $t\colon V/H \xrightarrow{\sim} \mathbf{C}$. Given such a triple (H, s, t) consider a map of sets

$$V \smallsetminus H \longrightarrow H, \quad v \mapsto \frac{s(v)}{t([v])}.$$

(c) Show that this map induces a homeomorphism $\mathbf{P}(V \smallsetminus H) \xrightarrow{\sim} H$.

(d) Prove that the collection of maps $\mathbf{P}(V \smallsetminus H) \xrightarrow{\sim} H$ defined by all possible triples (H, s, t) forms a complex manifold atlas on $\mathbf{P}(V)$. In view of (a) and (b) it remains to show that all the transition maps are biholomorphic.

Exercise 11. (2 *points*) Show that the Riemann surfaces S^2 and $\mathbf{P}(\mathbf{C}\oplus\mathbf{C})$ are isomorphic.

Exercise 12. (1 *point*) Consider the complex manifold $\mathbf{C}^2 := \mathbf{C}\times\mathbf{C}$ and let U be the open subset $\mathbf{C}^2 \smallsetminus \{0\}$. Let $f\colon U \to \mathbf{C}$ be a holomorphic function such that for every $\alpha \in \mathbf{C}^\times$ and every $z \in U$ we have $f(\alpha z) = f(z)$. Prove that f is constant.

Recall that a *lattice* $\Lambda \subset \mathbf{C}$ is a discrete and cocompact subgroup (of the additive group of $\mathbf{C}$). Every lattice is isomorphic to $\mathbf{Z} \oplus \mathbf{Z}$ as an abelian group. Consider the quotient $E := \mathbf{C}/\Lambda$ and let $\pi\colon \mathbf{C} \twoheadrightarrow E$ be the projection. Recall that we defined the structure sheaf $\mathcal{O}_E$ as follows:

$$\mathcal{O}_E(U) = \{\, f\colon U \longrightarrow \mathbf{C} \mid f \circ (\pi|_{\pi^{-1}(U)}) \text{ is holomorphic} \,\}.$$

Riemann surfaces of the form $E = \mathbf{C}/\Lambda$ are known as *elliptic curves*.

Exercise 13. (2 *points* for parts (a) and (b), and 2 *more points* for part (c))

(a) Let $\Lambda, \Lambda' \subset \mathbf{C}$ be lattices and let $E := \mathbf{C}/\Lambda$ and $E' := \mathbf{C}/\Lambda'$ be the corresponding quotients with the quotient maps π and π'. Let $f\colon E \to E'$ be a continuous map such that $f(0) = 0$. Show that f is a morphism of Riemann surfaces if and only if there is a (necessarily unique) complex number α such that the following diagram commutes:

$$\begin{tikzcd} \mathbf{C} \arrow[r, "z \mapsto \alpha z"] \arrow[d, "\pi"'] & \mathbf{C} \arrow[d, "\pi'"] \\ E \arrow[r, "f"] & E'. \end{tikzcd}$$

Here the vertical arrows are the projections.

Hint: construct a continuous map $\tilde{f}\colon \mathbf{C} \to \mathbf{C}$ which makes this square commutative ("lifts f"), and then use the compactness of the quotients to prove that $\tilde{f}$ has the desired form.

(b) Let E, E' be as above, and let $f\colon E \to E'$ be a morphism of Riemann surfaces. Prove that there is an automorphism $g\colon E' \to E'$ such that $g \circ f$ maps 0 to 0.

(c) Show that there are infinitely many lattices $\Lambda \subset \mathbf{C}$ such that the corresponding quotients $\mathbf{C}/\Lambda$ are pairwise nonisomorphic as Riemann surfaces.