

Exercise set 9

As in the lectures we denote by Ab the category of abelian groups and by Ab_X the category of sheaves (of abelian groups) on a topological space X . For each topological space X and each abelian group A we denote by $H^i(X, A)$ the i -th sheaf cohomology group of X with coefficients in the constant sheaf $\underline{A}_X$.

Exercise 1. Let X be a not necessarily compact Riemann surface. Show that the Laplace operator $\Delta = 2i\partial\bar{\partial}: \mathcal{A}_X^0 \rightarrow \mathcal{A}_X^2$ is an epimorphism of sheaves. (Hint: This is easier than it may look at a first glance.) Deduce the following local version of the Main Theorem:

Let η be a 2-form on X . Then for each point $x \in X$ there is an open neighbourhood U of x and a smooth function $f: U \rightarrow \mathbf{C}$ such that $\Delta f = \eta|_U$.

(The exactness condition on η is implicit in this statement because every form is locally exact by Poincaré's lemma.)

Exercise 2. Let $U \subset \mathbf{C}$ be a contractible open subset and let $f: U \rightarrow \mathbf{C} \setminus \{0\}$ be a holomorphic function without zeroes. Show that f is the composite $e \circ g$ of the exponential map $e: \mathbf{C} \rightarrow \mathbf{C} \setminus \{0\}$ and a holomorphic function $g: U \rightarrow \mathbf{C}$. (Hint: Have a look at Exercise 7(c) of Set 5.) Extra question: Does the claim hold when U is the punctured open unit disk around 0?

Here are some more exercises on sheaf cohomology:

Exercise 3. Let X be a Hausdorff second countable locally compact topological space. Let $\mathcal{C}_X$ be the sheaf of continuous real-valued functions on X . Prove that $H^i(X, \mathcal{C}_X) = 0$ for all $i \geq 1$. Hint: You can use without proof the fact that X is normal, so Urysohn's lemma applies.

Exercise 4. Let X be a topological space. Recall that a sheaf $\mathcal{F}$ on X is *flasque* if the restriction morphism $\mathcal{F}(X) \rightarrow \mathcal{F}(U)$ is surjective for every open subset $U \subset X$.

(a) Let $\mathcal{F}$ be a flasque sheaf on X . Show that $H^i(X, \mathcal{F}) = 0$ for all $i \geq 1$. Hints:

- Try to imitate the proof of Theorem 10.6 from the lectures. Work out analogs of Lemmas A and B, and combine them to show that $H^i(X, \mathcal{F}) = 0$.
- In the lectures we demonstrated that every injective sheaf is flasque. You are free to quote this result.
- In the analog of Lemma A you will not need the shrinking of covers and other complications that plagued the lectures. On the downside you will have to handle covers of infinite cardinality. Zorn's lemma will be of use.

Next, we denote by $X^\sharp$ the set X with the discrete topology and by $\iota: X^\sharp \rightarrow X$ the identity map of the underlying sets.

- (b) For each sheaf $\mathcal{F}$ on X prove that the adjunction unit $\mathcal{F} \hookrightarrow \iota_* \iota^{-1} \mathcal{F}$ is an injection, and that the sheaf $\iota_* \iota^{-1} \mathcal{F}$ is flasque.
- (c) Iterating this construction produce a functor $Ab_X \rightarrow \text{Ch}(Ab_X)$ that sends a sheaf $\mathcal{F}$ to its resolution by flasque sheaves. This is called the *Godement resolution* of $\mathcal{F}$.

By Acyclic Resolution Theorem of the lectures the Godement resolutions can be used to calculate cohomology of arbitrary sheaves. (This approach is not always practical.)

Here are some more exercises on homological algebra. We work with an abstract abelian category $\mathcal{A}$ although in each case you are free to suppose that $\mathcal{A}$ is the category of (left or right) modules over an associative ring.

Let $A^\bullet$ be a complex of objects in $\mathcal{A}$ and $n \in \mathbf{Z}$ an integer. In the following we denote by $A^\bullet[n]$ the *shifted complex* that is defined by the formula

$$(A^\bullet[n])^i := A^{i+n}, \quad d_{A^\bullet[n]}^i := (-1)^n d: A^{i+n} \rightarrow A^{i+n+1}.$$

This is the standard convention in the literature, but note that Weibel's book uses the opposite direction of shift.

Exercise 5. A short exact sequence of complexes

$$(*) \quad 0 \longrightarrow A^\bullet \longrightarrow B^\bullet \longrightarrow C^\bullet \longrightarrow 0$$

is *termwise split* if each individual sequence of objects $0 \rightarrow A^n \rightarrow B^n \rightarrow C^n \rightarrow 0$ is split. (Importantly, this is much weaker than demanding that the sequence $(*)$ is split in the category of complexes.)

- (a) Assuming that the sequence $(*)$ is termwise split pick splitting morphisms $s^n: C^n \rightarrow B^n$ and $t^n: B^n \rightarrow A^n$. Show that the family of morphisms

$$t^{n+1} \circ d_B^n \circ s^n: C^n \rightarrow A^{n+1}$$

defines a morphism of complexes $\delta: C^\bullet \rightarrow A^\bullet[1]$. Hint: try composing with suitable monomorphisms.

- (b) Show that the homotopy class of δ is independent of the choice of splittings s^n, t^n .
- (c) Show that the induced morphism $H^n(\delta): H^n(C^\bullet) \rightarrow H^n(A^\bullet[1]) = H^{n+1}(A^\bullet)$ is exactly the boundary homomorphism in the long exact sequence of cohomology induced by the short exact sequence $(*)$

This gives an alternative construction of the boundary homomorphism in the long exact sequence of cohomology.

Exercise 6. Let $A^\bullet$ and $B^\bullet$ be complexes of objects in $\mathcal{A}$. We define the *Hom complex* $\text{Hom}(A^\bullet, B^\bullet)^\bullet$ of abelian groups as follows. The objects of this complex are given by the formula

$$\text{Hom}(A^\bullet, B^\bullet)^i = \prod_{n \in \mathbf{Z}} \text{Hom}_{\mathcal{A}}(A^n, B^{n+i}).$$

An element of the abelian group $\text{Hom}(A^\bullet, B^\bullet)^i$ can be seen as a “termwise morphism” $f: A^\bullet \rightarrow B^\bullet[i]$ that need not commute with the differentials, and so is not necessarily a morphism of complexes. The differentials of A and B are “termwise morphisms” $d_A: A^\bullet \rightarrow A^\bullet[1]$ and $d_B: B^\bullet \rightarrow B^\bullet[1]$. The differential of the Hom complex is defined by the formula:

$$d^i(f) := d_B \circ f - (-1)^i f \circ d_A.$$

- (a) Verify that $\text{Hom}(A^\bullet, B^\bullet)^\bullet$ is a complex of abelian groups, i.e. the square of the differential is zero.
- (b) Show that for each integer n there is an isomorphism

$$H^n(\text{Hom}(A^\bullet, B^\bullet)^\bullet) \xrightarrow{\sim} \text{Hom}_{K(\mathcal{A})}(A^\bullet, B^\bullet[n])$$

that is natural in $A^\bullet$ and $B^\bullet$.

The Hom complex is useful when one needs to control Hom sets in the homotopy category $K(\mathcal{A})$. To tie this up with the previous exercise, here are a few extra questions:

- (c) Let $T^\bullet$ be a complex in $\mathcal{A}$. Show that applying the functor $\text{Hom}(T^\bullet, \)^\bullet$ to the termwise split sequence $(*)$ we obtain a termwise split sequence

$$0 \longrightarrow \text{Hom}(T^\bullet, A^\bullet)^\bullet \longrightarrow \text{Hom}(T^\bullet, B^\bullet)^\bullet \longrightarrow \text{Hom}(T^\bullet, C^\bullet)^\bullet \longrightarrow 0.$$

- (d) In the special case $T^\bullet = C^\bullet$ we have a sequence

$$0 \longrightarrow \text{Hom}(C^\bullet, A^\bullet)^\bullet \longrightarrow \text{Hom}(C^\bullet, B^\bullet)^\bullet \longrightarrow \text{Hom}(C^\bullet, C^\bullet)^\bullet \longrightarrow 0.$$

This induces a long exact cohomology sequence, so by part (b) we get a boundary homomorphism

$$\text{Hom}_{K(\mathcal{A})}(C^\bullet, C^\bullet) \longrightarrow \text{Hom}_{K(\mathcal{A})}(C^\bullet, A^\bullet[1]).$$

Calculate the image of the identity $1_{C^\bullet}$ under this homomorphism.

Exercise 7. Let A and B be objects of $\mathcal{A}$. Recall that an *extension* of A by B is a short exact sequence of the form $0 \rightarrow B \rightarrow E \rightarrow A \rightarrow 0$. A morphism from an extension $\mathcal{E} = (0 \rightarrow B \rightarrow E \rightarrow A \rightarrow 0)$ to an extension $\mathcal{E}' = (0 \rightarrow B \rightarrow E' \rightarrow A \rightarrow 0)$ is a morphism $f: E \rightarrow E'$ which makes the following diagram commutative:

$$\begin{array}{ccccccc} 0 & \longrightarrow & B & \begin{array}{c} \nearrow \\ \searrow \end{array} & \begin{array}{c} E \\ \downarrow f \\ E' \end{array} & \begin{array}{c} \nearrow \\ \searrow \end{array} & A \longrightarrow 0. \end{array}$$

We have seen (Exercise 11(a) of Set 8) that f is necessarily an isomorphism. In this exercise we will classify extensions of A by B via a suitable cohomology group.

To this end, consider the functor $h_A := \text{Hom}_{\mathcal{A}}(A, \): \mathcal{A} \rightarrow \text{Ab}$. We have seen that this functor is left exact (Exercise 2 of Set 5). Suppose that the category $\mathcal{A}$ has enough injectives, so that the right derived functors $R^n h_A$ are defined. We set:

$$\text{Ext}^n(A, \) := R^n h_A(\).$$

We will prove that isomorphism classes of extensions of A by B are in bijection with the elements of the abelian group $\text{Ext}^1(A, B)$.

As usual you are free to assume that $\mathcal{A}$ is the category of left or right modules over a ring. You will benefit from the naturality of long exact sequences of derived functors. This property was claimed without proof in the lectures (and the proof is somewhat subtle).

- (a) Let $\mathcal{E} = (0 \rightarrow B \rightarrow E \rightarrow A \rightarrow 0)$ be an extension. This is in particular a short exact sequence, so applying the functor h_A we get a long exact sequence of cohomology. The first boundary homomorphism in it has the form

$$\begin{array}{ccc} h_A(A) & \xrightarrow{\delta} & R^1 h_A(B) \\ \parallel & & \parallel \\ \text{Hom}(A, A) & & \text{Ext}^1(A, B) \end{array}$$

Set $c_{\mathcal{E}} := \delta(1_A)$ and show that the cohomology class $c_{\mathcal{E}} \in \text{Ext}^1(A, B)$ depends only on the isomorphism class of $\mathcal{E}$.

- (b) Prove that for every cohomology class $c \in \text{Ext}^1(A, B)$ there is an extension $\mathcal{E} = (0 \rightarrow B \rightarrow E \rightarrow A \rightarrow 0)$ such that $c_{\mathcal{E}} = c$.

You can approach this as follows. Pick an embedding $B \hookrightarrow I$ to an injective object. Show that the short exact sequence $0 \rightarrow B \rightarrow I \rightarrow I/B \rightarrow 0$ induces a long exact sequence with a surjective boundary homomorphism

$$\text{Hom}(A, I/B) \xrightarrow{\delta} \text{Ext}^1(A, B).$$

Given a cohomology class $c \in \text{Ext}^1(A, B)$ pick a morphism $f: A \rightarrow I/B$ such that $\delta(f) = c$. Denoting by $q: I \rightarrow I/B$ the quotient morphism consider the object

$$E := \ker(A \oplus I \xrightarrow{f-q} I/B).$$

Show that the natural projection $E \rightarrow A$ is an epimorphism. Prolong this to an extension $\mathcal{E} = (0 \rightarrow B \rightarrow E \rightarrow A \rightarrow 0)$ such that $c_{\mathcal{E}} = c$.

- (c) Show that extensions $\mathcal{E}$ and $\mathcal{E}'$ are isomorphic if and only if $c_{\mathcal{E}} = c_{\mathcal{E}'}$. Hint: The construction of (b) gives you an extension $\mathcal{E}''$ such that $c_{\mathcal{E}''} = c_{\mathcal{E}}$. Use the universal property of kernels to show that $\mathcal{E}$ is isomorphic to $\mathcal{E}''$.
- (d) Deduce that an extension $\mathcal{E}$ splits if and only if $c_{\mathcal{E}} = 0$.