

Exercise set 7

As in the lectures we denote by Ab the category of abelian groups, by Ab_X the category of sheaves (of abelian groups) on a topological space X , and by PAb_X the category of presheaves of abelian groups on X . For each topological space X and each abelian group A we denote by $H^i(X, A)$ the i -th sheaf cohomology group of X with coefficients in the constant sheaf $\underline{A}_X$.

First, let us check the understanding of some material from the lectures:

Exercise 1. Let $f: X \rightarrow Y$ be a continuous map of topological spaces. Suppose that f is a homotopy equivalence. Show that for any $i \geq 0$ and any abelian group A the pullback morphism $f^*: H^i(Y, A) \xrightarrow{\sim} H^i(X, A)$ on the respective sheaf cohomology groups is an isomorphism. (Recall that a continuous map $f: X \rightarrow Y$ is a homotopy equivalence if and only if there is a continuous map $g: Y \rightarrow X$ such that the composites $f \circ g$ and $g \circ f$ are homotopy equivalent to the respective identity maps.)

Next, we move to cohomology of sheaves with constant coefficients.

Exercise 2. (*The Mayer–Vietoris sequence*) Let X be a topological space and let Y, Z be closed subsets of X such that $Y \cup Z = X$. Let also A be an abelian group. Prove that there is a long exact sequence of sheaf cohomology groups of the form

$$\dots \longrightarrow H^i(X, A) \xrightarrow{e} H^i(Y, A) \oplus H^i(Z, A) \xrightarrow{e'} H^i(Y \cap Z, A) \longrightarrow H^{i+1}(X, A) \longrightarrow \dots$$

where

- the arrow e is given by the pullback morphisms that are induced by the inclusions $Y \hookrightarrow X$ and $Z \hookrightarrow X$;
- the arrow e' is the difference of the pullback morphisms that are induced by the inclusions $Y \cap Z \hookrightarrow Y$ and $Y \cap Z \hookrightarrow Z$.

The following exercise repeats the one that we had in the take-home assignment:

Exercise 3. For each $n \geq 2$ let S^n be the topological n -dimensional sphere. Calculate the sheaf cohomology groups $H^i(S^n, \mathbf{Z})$ for all $n \geq 2$ and $i \geq 0$. Namely, show that:

$$H^i(S^n, \mathbf{Z}) \cong \begin{cases} \mathbf{Z} & i \in \{0, n\}, \\ 0 & i \notin \{0, n\}. \end{cases}$$

Here you can invoke Exercise 2 if you did it.

The next exercise is trickier than Exercise 3, but it can be done by the same method:

Exercise 4. Let $T := S^1 \times S^1$ be the topological torus. Calculate the sheaf cohomology groups $H^i(T, \mathbf{Z})$ for all i . Namely, prove that

$$H^i(T, \mathbf{Z}) \cong \begin{cases} \mathbf{Z}, & i \in \{0, 2\}, \\ \mathbf{Z} \oplus \mathbf{Z}, & i = 1 \\ 0, & i > 2. \end{cases}$$

Here you can invoke Exercise 2 if you did it. Hint: Exercise 1 should be useful here.

The next exercise is harder still, but it can also be solved by the same method as Exercise 4:

Exercise 5. Let X be a compact orientable topological surface of genus g . Calculate the sheaf cohomology groups $H^i(X, \mathbf{Z})$ for all i . Namely, show that

$$H^i(X, \mathbf{Z}) \cong \begin{cases} \mathbf{Z}, & i \in \{0, 2\}, \\ \mathbf{Z}^{2g}, & i = 1 \\ 0, & i > 2. \end{cases}$$

Again, you are free to invoke Exercise 2 if you did it.

Next, let us review the pullback maps on sheaf cohomology in some detail:

Exercise 6. In this exercise we will recall the construction of the pullback map. We will also prove its compatibility with compositions, thus finishing the proof of the homotopy invariance theorem. Let us fix an abelian group A .

- (a) Let $f: X \rightarrow Y$ be a continuous map. Show that the inverse image sheaf $f^{-1}(\underline{A}_Y)$ is isomorphic to $\underline{A}_X$. Hint: first consider the case when Y is a point, and use the compatibility of inverse images with compositions.
- (b) Use a suitable adjunction to construct for every continuous map $f: X \rightarrow Y$ an isomorphism $\mu_f: f^{-1}(\underline{A}_Y) \xrightarrow{\sim} \underline{A}_X$ with the following compatibility property: For every continuous map $g: X' \rightarrow X$ we have an equality

$$\mu_g \circ g^{-1}(\mu_f) = \mu_{fg}.$$

Now recall the definition of the pullback morphisms $f^*: H^i(Y, \underline{A}_Y) \rightarrow H^i(X, \underline{A}_X)$. Let $\mathcal{I}_X^\bullet$ be an injective resolution of $\underline{A}_X$ (that is defined uniquely up to homotopy), and likewise let $\mathcal{I}_Y^\bullet$ be an injective resolution of $\underline{A}_Y$. Via the isomorphism μ_f we view $\mathcal{I}_X^\bullet$ as an injective resolution of $f^{-1}(\underline{A}_Y)$. By adjunction $f^{-1} \leftrightarrow f_*$ we get a morphism of complexes $\underline{A}_Y[0] \rightarrow f_*\mathcal{I}_X^\bullet$. By the universal property of injective resolutions (Theorem 7.8) this morphism factors uniquely (up to homotopy) through the resolution morphism $\underline{A}_Y[0] \rightarrow \mathcal{I}_Y^\bullet$. We thus obtain a well-defined morphism

$$\mathcal{I}_Y^\bullet \longrightarrow f_*\mathcal{I}_X^\bullet.$$

Applying the global sections functor $\Gamma(Y, \cdot): \mathcal{F} \mapsto \mathcal{F}(Y)$ and taking the i -th cohomology group we get the pullback morphism

$$f^*: H^i(Y, \underline{A}_Y) \longrightarrow H^i(X, \underline{A}_X).$$

Here we are quietly using the fact that $\Gamma(Y, f_*\mathcal{F}) = \Gamma(X, \mathcal{F})$ for any sheaf $\mathcal{F}$ on X .

- (c) Deduce from (b) that for each pair of composable continuous maps $f: X \rightarrow Y$ and $g: X' \rightarrow X$ the pullback morphisms satisfy

$$g^* \circ f^* = (fg)^*.$$

Hint: Use the naturality of adjunctions and the universal property of Theorem 7.8.

Exercise 7. Let $f: M \rightarrow N$ be a morphism of smooth manifolds. By calculus on manifolds we have a pullback morphism on differential i -forms for every $i \geq 0$:

$$f^*: \Gamma(N, \mathcal{F}_N^i) \longrightarrow \Gamma(M, \mathcal{F}_M^i).$$

- (a) Show that this induces a well-defined morphism of de Rham cohomology groups:

$$f_{\text{dR}}^*: H_{\text{dR}}^i(N, \mathbf{R}) \longrightarrow H_{\text{dR}}^i(M, \mathbf{R})$$

Next recall how the de Rham (iso)morphism is defined. In the lectures we deduced from Poincaré lemma that the de Rham complex $\mathcal{F}_M^\bullet$ is a resolution of $\underline{\mathbf{R}}_M$. Let $\mathcal{I}^\bullet$ be an injective resolution of $\underline{\mathbf{R}}_M$. By the universal property of resolutions in Theorem 7.7 the morphism of complexes $\underline{\mathbf{R}}_M[0] \rightarrow \mathcal{I}^\bullet$ factors uniquely (up to homotopy) through the resolution morphism $\underline{\mathbf{R}}_M[0] \rightarrow \mathcal{F}_M^\bullet$. We thus obtain a well-defined morphism

$$\mathcal{F}_M^\bullet \longrightarrow \mathcal{I}^\bullet$$

in the homotopy category $K(\text{Ab}_M)$. Applying the global sections functor and taking the cohomology groups we obtain for each i a morphism

$$H_{\text{dR}}^i(M, \mathbf{R}) \longrightarrow H^i(M, \underline{\mathbf{R}}_M).$$

This is the *de Rham morphism*, and in the lectures we proved that it is an isomorphism for every smooth manifold M .

- (b) Show that we have a commutative square

$$\begin{array}{ccc} H_{\text{dR}}^i(N, \mathbf{R}) & \xrightarrow{f_{\text{dR}}^*} & H_{\text{dR}}^i(M, \mathbf{R}) \\ \downarrow \wr & & \downarrow \wr \\ H^i(N, \mathbf{R}) & \xrightarrow{f^*} & H^i(M, \mathbf{R}) \end{array}$$

where the vertical arrows are de Rham isomorphisms and the bottom arrow is the pullback of sheaf cohomology classes as defined in Exercise 6. Hint: Extend f_{dR}^* to a morphism of complexes $\mathcal{F}_N^\bullet \rightarrow f_* \mathcal{F}_M^\bullet$ and use the universal property of Theorem 7.7.

Exercise 8. To complete the proof of de Rham's theorem let us review the *cone construction*. Let $f: A^\bullet \rightarrow B^\bullet$ be a morphism of complexes of objects in an abelian category $\mathcal{A}$. The *cone of the morphism* f is a complex $C^\bullet$ defined by the formula

$$C^n := A^{n+1} \oplus B^n \quad \text{with the differential} \quad d_C^n := \begin{pmatrix} -d_A^{n+1} & 0 \\ f^n & d_B^n \end{pmatrix}.$$

When $\mathcal{A}$ is the category of (left or right) modules over a ring, we have the following formula for the action of d_C on elements:

$$d_C^n(a, b) = (-d_A^{n+1}(a), f^n(a) + d_B^n(b)).$$

(a) Check that $C^\bullet$ is indeed a complex, i.e. $d_C^{n+1} \circ d_C^n = 0$.

Next, let the complex $A^\bullet[1]$ be defined by the formula

$$(A^\bullet[1])^n := A^{n+1}, \quad d_{A^\bullet[1]}^n := -d_A^{n+1}.$$

(b) Show that we have a well-defined short exact sequence of complexes

$$0 \longrightarrow B^\bullet \longrightarrow C^\bullet \longrightarrow A^\bullet[1] \longrightarrow 0$$

where the morphism $B^\bullet \rightarrow C^\bullet$ is given by the inclusions $B^n \hookrightarrow C^n$ and the morphism $C^\bullet \rightarrow A^\bullet[1]$ is given by the projections $C^n \twoheadrightarrow A^{n+1}$.

(c) The short exact sequence of (b) induces a long exact sequence of cohomology objects. By construction we have $H^{n-1}(A^\bullet[1]) = H^n(A^\bullet)$ so the boundary homomorphisms take the form

$$H^n(A^\bullet) \xrightarrow{\delta} H^n(B^\bullet).$$

Prove that these homomorphisms coincide with $H^n(f)$. Here you are free to either invoke Freyd–Mitchell theorem or assume that the abelian category $\mathcal{A}$ is the category of (left or right) modules over a ring.