Exercise set 6

As in the lectures we denote by Ab the category of abelian groups, by Ab_X the category of sheaves (of abelian groups) on a topological space X, and by PAb_X the category of presheaves of abelian groups on X.

Let us recap some material of the lectures. First, some examples of adjoint functors:

Exercise 1. Let X be a topological space. Demonstrate that the sheafification functor $(\ )^{\#}\colon\ PAb_X \to Ab_X$ is left adjoint to the inclusion functor $\iota\colon\ Ab_X \hookrightarrow PAb_X$.

Exercise 2. Let Grp be the category of groups, and let $U\colon\ Grp \to Sets$ be the functor that sends a group to its set of elements. Show that U admits a left adjoint functor $F\colon\ Sets \to U$.

Exercise 3. Every homomorphism of rings $R \to S$ gives rise to a pair of functors: The functor of extension of scalars

$$M \mapsto S \otimes_R M$$

from left R-modules to left S-modules, and the functor of restriction of scalars

$$N \mapsto N|_R^S$$

from left S-modules to left R-modules. The left R-module $N|_R^S$ has the underlying additive group N with R acting through the S-module structure of N.

(a) Show that the functors $S \otimes_R (\ )$ and $(\ )\big|_R^S$ are adjoint. More precisely, construct a natural isomorphism

$$\mathrm{Hom}_S(S \otimes_R M,\, N) \;\cong\; \mathrm{Hom}_R(M,\, N|_R^S).$$

(b) Deduce that the functor $S \otimes_R (\ )$ is right exact.

(c) For each left R-module M write down a formula for the unit of adjunction

$$M \;\longrightarrow\; (S \otimes_R M)\big|_R^S.$$

Likewise, for each left S-module N write down a formula for the counit of adjunction

$$S \otimes_R (N|_R^S) \;\longrightarrow\; N.$$

Some more exercises on adjunction:

Exercise 4. Show that for any adjunction $F \leftrightarrow G$ the following are equivalent:

(a) The right adjoint functor G is fully faithful.

(b) The counit of adjunction $FG \to 1$ is an isomorphism.

Next we study colimits.

Exercise 5. Let $f\colon A \to B$ be a morphism in an abelian category. Show that the colimit of the diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d] & B \\ 0 & \end{tikzcd}$$

is the cokernel of f.

Exercise 6. Let $\mathcal{I}$ be a small category, and let $D\colon \mathcal{I} \to Ab$ be a functor (what is called a diagram in the lectures).

(a) Prove that the object $\operatorname{colim} D$ exists. Hint: Consider the abelian group $\bigoplus_{i\in\mathcal{I}} D(i)$, and take an appropriate quotient.

(b) Assuming that $\mathcal{I}$ has a terminal object t show that $\operatorname{colim} D = D(t)$.

The next exercise is optional, and can be somewhat tricky:

Exercise 7. Let $\mathcal{I}$ be a small category, and consider the category $[\mathcal{I}, Ab]$ of functors $\mathcal{I} \to Ab$.

(a) Show that the category $[\mathcal{I}, Ab]$ is abelian with kernels and cokernels computed in the termwise manner.

(b) Show that the functor $\operatorname{colim}\colon [\mathcal{I}, Ab] \to Ab$ is right exact.

(c) Assuming that the category $\mathcal{I}$ is filtered prove that the functor $\operatorname{colim}\colon [\mathcal{I}, Ab] \to Ab$ is exact.

Recall that $\mathcal{I}$ is filtered it satisfies the following two conditions:

- For every pair of objects $i, i' \in \mathcal{I}$ there is an object j and arrows $i \to j$, $i' \to j$.
- For every pair of morphism $f, g\colon i \to i'$ there is a morphism $h\colon i' \to j$ such that $h \circ f = h \circ g$.

Next, we move to sheaves.

Exercise 8. Let X be a topological space. In this exercise we will show that every injective sheaf on X is flasque, as was claimed in the lecture. Let $U \subset X$ be an open subset, and let $\underline{\mathbf{Z}}_{X,U}$ be the sheafification of the presheaf

$$P_{X,U}\colon V \ \longmapsto \ \begin{cases} \mathbf{Z}, & V \subset U, \\ 0, & V \not\subset U. \end{cases}$$

Note that $\underline{\mathbf{Z}}_{X,X}$ is the constant sheaf $\underline{\mathbf{Z}}_X$.

(a) For each sheaf $\mathcal{F}$ on X construct a natural isomorphism

$$\mathcal{F}(U) \ \cong \ \operatorname{Hom}(\underline{\mathbf{Z}}_{X,U}, \mathcal{F}).$$

This generalizes Exercise 4 of Set 5.

(*b*) Let $\mu\colon P_{X,U} \to P_{X,X}$ be the morphism of presheaves such that $\mu_V = 1$ when $V \subset U$ and $\mu_V = 0$ otherwise. Show that the induced morphism of sheaves $\underline{\mathbf{Z}}_{X,U} \to \underline{\mathbf{Z}}_X$ is a monomorphism. Hint: Consider stalks.

(*c*) Show that we have a commutative square

$$\begin{tikzcd}
\operatorname{Hom}(\underline{\mathbf{Z}}_X, \mathcal{F}) \arrow[r, "(b)"] \arrow[d, "(a)"'] & \operatorname{Hom}(\underline{\mathbf{Z}}_{X,U}, \mathcal{F}) \arrow[d, "(a)"] \\
\mathcal{F}(X) \arrow[r, "\text{restriction}"] & \mathcal{F}(U).
\end{tikzcd}$$

(*d*) Now let $\mathcal{I}$ be an injective sheaf. Deduce that for every open $U \subset X$ the morphism $\mathcal{I}(X) \twoheadrightarrow \mathcal{I}(U)$ is surjective, and concude that $\mathcal{I}$ is flasque.

Here are some more exercises on homological algebra:

Exercise 9. Let R be a commutative integral domain, and let M be an R-module. Suppose that M is an injective object in the category of R-modules. Show that M is *divisible*: For every $m \in M$ and every nonzero $r \in R$ there exists an element $m' \in M$ such that $rm' = m$.

Exercise 10. Let R be a ring and let M be a right R-module. Prove that the functor $N \mapsto M \otimes_R N$ from left R-modules to abelian groups is right exact.

Exercise 11. Let k be a field. Consider the commutative ring $R := k[X]/(X^2)$, the R-module $M := R/(X)$ and the functor $F\colon N \mapsto M \otimes_R N$ from R-modules to abelian groups (or R-modules, as this will make no difference). This functor is right exact by the exercise above. Let $L^nF(\ )$ be its left derived functors. Compute $L^nF(M)$ for all $n \geqslant 0$.

Exercise 12. Let k be a field. A *filtered k-vector space* $X = (V, (V_i)_{i\in\mathbf{Z}})$ is a k-vector space V equipped with an increasing filtration by k-vector subspaces:

$$\cdots \subseteq V_{i-1} \subseteq V_i \subseteq V_{i+1} \subseteq \dots$$

Let $Y = (W, (W_i)_{i\in\mathbf{Z}})$ be another filtered k-vector space. A morphism of filtered vector spaces $f : X \to Y$ is a k-linear map $f : V \to W$ such that $f(V_i) \subseteq W_i$ for all i. Filtered k-vector spaces form a category denoted by FVec_k.

(*a*) Show that FVec_k is additive.

(*b*) Show that each morphism in FVec_k has a kernel and a cokernel.

(*c*) Show that there is a morphism in FVec_k that has zero kernel and cokernel, but is not itself an isomorphism (i.e. is not invertible). Hint: Pick a nonzero vector space with a nontrivial filtration, and try to “shift” the numbering in the filtration.

Conclude that the category FVec_k is not abelian.