Exercise set 4

Exercise 1. Let $\mathcal{A}$ and $\mathcal{B}$ be additive categories, and let $F\colon \mathcal{A} \to \mathcal{B}$ be a functor.

(a) For each pair of objects $A, A' \in \mathcal{A}$ construct a morphism
$$F(A \oplus A') \longrightarrow F(A) \oplus F(A')$$
that is natural in A and A'.

(b) Show that the morphism in (a) is an isomorphism if and only if the functor F is additive, i.e. preserves sums of morphisms.

Next, here are some exercises with sheaves. Let us begin with sheaves on a general topological space X.

Exercise 2. Show that, as was claimed in the lecture, the sheafification $\mathcal{F}^{\#}$ is a sheaf.

Exercise 3. Let A be an abelian group and let X be a topological space. Consider a presheaf $P_{A,X}$ defined by the formula $P_{A,X}(U) := A$. Consider also a sheaf $\underline{A}_X$ defined by the formula
$$\underline{A}_X(U) := \{ \text{ locally constant maps } f\colon U \to A \}.$$
(Recall that a map is *locally constant* if it is continuous w.r.t. the discrete topology on A.) Construct an isomorphism of sheaves
$$P^{\#}_{A,X} \xrightarrow{\sim} \underline{A}_X.$$

Remark. The sheaf $\underline{A}_X$ is called the *constant sheaf* with values in A.

Exercise 4. Let $\phi\colon \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves on a topological space X. Consider a presheaf $\mathcal{H}$ defined by the formula
$$\mathcal{H}(U) := \mathrm{Image}\big(\phi_U\colon \mathcal{F}(U) \longrightarrow \mathcal{G}(U)\big).$$
Show that the sheafification $\mathcal{H}^{\#}$ is naturally isomorphic to $\ker(\mathrm{coker}(\phi))$. Hint: Construct a natural isomorphism $\mathrm{coker}(\ker(\phi)) \xrightarrow{\sim} \mathcal{H}^{\#}$ and use Exercise 7 (a) of Exercise set 3.

Now we consider sheaves on the closed unit interval $[0,1]$.

Exercise 5. Let X be the closed unit interval $[0,1]$. Consider a presheaf $\mathcal{F}$ on X defined by the formula
$$\mathcal{F}(U) := \{ \text{ bounded continuous functions } f\colon U \to \mathbf{C} \}.$$
Is this presheaf a sheaf?

Exercise 6. Let X be the closed interval $[0, 1]$.

(a) Show that up to isomorphism there is a unique sheaf $\mathcal{F}$ on X with stalks $\mathcal{F}_0 = \mathcal{F}_1 = \mathbf{Z}$ and $\mathcal{F}_x = 0$ for all $x \in (0, 1)$.

(b) For each abelian group A calculate the Hom groups

$$\mathrm{Hom}(\mathcal{F},\, \underline{A}_X) \quad \text{and} \quad \mathrm{Hom}(\underline{A}_X,\, \mathcal{F}).$$

The next exercise may be somewhat hard at this stage:

Exercise 7. Let X be the closed unit interval $[0, 1]$. Does there exist a sheaf $\mathcal{F}$ on X such that $\mathcal{F}_x = 0$ at $x = \frac{1}{2}$ but $\mathcal{F}_x \neq 0$ at all other points x?

Finally, let us return to Riemann surfaces.

Exercise 8. Let X be a Riemann surface. Recall that $\mathcal{O}_X$ denotes the *structure sheaf* of X, i.e. the sheaf of holomorphic functions. Pick a point $x \in X$, and consider the stalk $\mathcal{O}_{X,x}$. Since $\mathcal{O}_X$ is a sheaf of commutative rings, the stalk $\mathcal{O}_{X,x}$ is naturally a commutative ring.

Now pick a local coordinate z at x such that $z(x) = 0$.

(a) Let $\mathbf{C}[[z]]$ be the ring of formal power series in z. Show that the power series expansion with respect to z induces an injective homomorphism of rings $\mathcal{O}_{X,x} \hookrightarrow \mathbf{C}[[z]]$.

(b) Prove that $\mathcal{O}_{X,x}$ is a discrete valuation ring with uniformizer z and residue field $\mathbf{C}$.

(c) (Extra) Show that the image of $\mathcal{O}_{X,x}$ in $\mathbf{C}[[z]]$ does not depend on the choices of z and x. Hint: You need to characterize the image in terms of coefficients of the power series expansion.