Exercise set 3

Here is one more exercise that I forgot to add to the first set:

Exercise 1. Let X, Y be Riemann surfaces, and let $f, g\colon X \to Y$ be morphisms thereof. Suppose that X is connected and that there is a nonempty open subset $U \subset X$ such that $f|_U = g|_U$. Prove that $f = g$. (You are free to use the fact that X is path-connected.)

Recall that a *lattice* $\Lambda \subset \mathbf{C}$ is a discrete and cocompact subgroup (of the additive group of $\mathbf{C}$). Every lattice is isomorphic to $\mathbf{Z} \oplus \mathbf{Z}$ as an abelian group. Consider the quotient $E := \mathbf{C}/\Lambda$ and let $\pi\colon \mathbf{C} \twoheadrightarrow E$ be the projection. Recall that we defined the structure sheaf $\mathcal{O}_E$ as follows:

$$\mathcal{O}_E(U) = \{\, f\colon U \longrightarrow \mathbf{C} \mid f \circ (\pi|_{\pi^{-1}(U)}) \text{ is holomorphic} \,\}.$$

Riemann surfaces of the form $E = \mathbf{C}/\Lambda$ are known as *elliptic curves.*

Exercise 2. (*a*) Let $\Lambda, \Lambda' \subset \mathbf{C}$ be lattices and let $E := \mathbf{C}/\Lambda$ and $E' := \mathbf{C}/\Lambda'$ be the corresponding quotients with the quotient maps π and π'. Let $f\colon E \to E'$ be a continuous map such that $f(0) = 0$. Show that f is a morphism of Riemann surfaces if and only if there is a (necessarily unique) complex number α such that the following diagram commutes:

$$\begin{tikzcd} \mathbf{C} \arrow[r, "z \mapsto \alpha z"] \arrow[d, "\pi"'] & \mathbf{C} \arrow[d, "\pi'"] \\ E \arrow[r, "f"] & E'. \end{tikzcd}$$

Here the vertical arrows are the projections.

Hint: construct a continuous map $\tilde{f}\colon \mathbf{C} \to \mathbf{C}$ which makes this square commutative ("lifts f"), and then use the compactness of the quotients to prove that $\tilde{f}$ has the desired form.

(*b*) Let E, E' be as above, and let $f\colon E \to E'$ be a morphism of Riemann surfaces. Prove that there is an automorphism $g\colon E' \to E'$ such that $g \circ f$ maps 0 to 0.

The next exericse is of increased difficulty:

Exercise 3. Show that there are infinitely many lattices $\Lambda \subset \mathbf{C}$ such that the corresponding quotients $\mathbf{C}/\Lambda$ are pairwise nonisomorphic as Riemann surfaces.

The compact topological space $S^1 \times S^1$ thus carries infinitely many distinct structures of a Riemann surface. This contrasts with the case of S^2 where such a structure is unique (as we will see later in the course).

Next, here are some exercises in category theory.

Exercise 4. Let k be a field. Consider the category $\mathcal{A}$ whose objects are integers $n \geqslant 0$, whose Hom sets $\mathrm{Hom}(n, m)$ consist of all $n \times m$-matrices with coefficients in k, and whose composition rule is given by the matrix multiplication. Prove that $\mathcal{A}$ is equivalent to the category of finite dimensional k-vector spaces.

Exercise 5. Let $\mathcal{A}$ be a category, and let $\mathcal{B}$ and $\mathcal{C}$ be small categories (i.e. categories in which the collection of objects is a set). We then have the category of functors $[\mathcal{A}, \mathcal{B}]$ from $\mathcal{A}$ to $\mathcal{B}$ with natural transformations as morphisms. In the same manner we have the category of functors $[\mathcal{A}, \mathcal{C}]$.

Let also $J\colon \mathcal{B} \to \mathcal{C}$ be a functor. The composition with J induces a functor

$$J \circ - \colon [\mathcal{A}, \mathcal{B}] \longrightarrow [\mathcal{A}, \mathcal{C}].$$

Assume that J is fully faithful.

(a) Prove that $J \circ -$ is fully faithful.

(b) Let $G, G' \in [\mathcal{A}, \mathcal{B}]$ be functors such that $J \circ G \cong J \circ G'$. Deduce that $G \cong G'$.

Let us move to additive and abelian categories. The following simple exercise is to check your understanding of the basics:

Exercise 6. Let $\mathcal{A}$ be an additive catgeory. Show that every kernel in $\mathcal{A}$ is a monomorphism and every cokernel in $\mathcal{A}$ is an epimorphism.

Here is a more elaborate exercise on kernels and cokernels:

Exercise 7. Let $f\colon A \to B$ be a morphism in an abelian category. The *image* $\mathrm{Im}(f)$ is defined to be the kernel of the cokernel of f. Dually, the *coimage* $\mathrm{Coim}(f)$ is the cokernel of the kernel of f.

(a) Show that there is a unique morphism $f'\colon \mathrm{Coim}(f) \longrightarrow \mathrm{Im}(f)$ that makes the following square commutative:

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d] & B \\ \mathrm{Coim}(f) \arrow[r, "f'"] & \mathrm{Im}(f). \arrow[u] \end{tikzcd}$$

(b) Show that f' is an isomorphism.

(c) Conclude that f factorizes into a composite of an epimorphism followed by a monomorphism.