Exercise set 1

Let us begin with some technical exercises:

Exercise 1. Let X be a topological space and let $\mathcal{A}$ and $\mathcal{B}$ be Riemann surface atlases on X. Prove that the atlases $\mathcal{A}$ and $\mathcal{B}$ are equivalent if and only if the identity map $1\colon (X, \mathcal{A}) \to (X, \mathcal{B})$ is a morphism of topological spaces equipped with Riemann surface atlases. In the lectures we used the "only if" direction. (Warning: This may be harder than it looks.)

Exercise 2. Let X be a Riemann surface and let $U \subset X$ be a nonempty open subset with the induced structure of a Riemann surface. Prove that this is the unique structure with respect to which the canonical map $U \hookrightarrow X$ is a morphism of Riemann surfaces.

Exercise 3. Construct an example of a non-Hausdorff topological space that admits a Riemann surface atlas.

Exercise 4. Fill in details of the discussion of tangent spaces in the lectures.

(a) Let W be a finite dimensional $\mathbf{R}$-vector space. Construct for every point $u \in W$ an isomorphism $T_u W \cong W$ that is canonical (in a sense you personally find satisfactory).

A Riemann surface X is in particular a smooth manifold of dimension 2, so for each point $x \in X$ we have a tangent space $T_x X$ that is an $\mathbf{R}$-vector space of dimension 2. A local coordinate $z\colon U \to \mathbf{C}$ around x induces an isomorphism $T_x(z)\colon T_x X \xrightarrow{\sim} T_{z(x)}\mathbf{C}$. By (a) the target is canonically isomorphic to $\mathbf{C}$, so by transport of structure the tangent space $T_x X$ becomes a $\mathbf{C}$-vector space of dimension 1.

(b) Prove that the resulting structure is independent of the choice of the local coordinate z.

(c) Let $U \subset \mathbf{C}$ be an open subset and let $f\colon U \to \mathbf{C}$ be a smooth map. Show that f is holomorphic if and only if for every point $x \in U$ the induced map $T_x(f)\colon T_x U \to T_x \mathbf{C}$ is $\mathbf{C}$-linear.

(d) Let X and Y be Riemann surfaces, and let $f\colon X \to Y$ be a smooth map. Deduce that f is a morphism of Riemann surfaces if and only if for every point $x \in X$ the induced map $T_x X \to T_{f(x)} Y$ is $\mathbf{C}$-linear.

Next we move to exercises that are more geometric in nature:

Exercise 5. Consider the open subset

$$U := \{x + iy \in \mathbf{C} \mid y > 0\}.$$

Show that U is isomorphic as a Riemann surface to the open unit disk Δ.

Exercise 6. Show that every rational function $f \in \mathbf{C}(X)$ gives rise to a morphism of Riemann surfaces $F\colon S^2 \to S^2$, in the following way:

Write f as a quotient $\frac{P}{Q}$ of coprime polynomials. At every point $z \in \mathbf{C}$ define

$$F(z) := \begin{cases} \frac{P(z)}{Q(z)} & \text{if } Q(z) \neq 0, \\ \infty & \text{otherwise.} \end{cases}$$

Define also

$$F(\infty) := \begin{cases} 0 & \text{if } \deg(P) < \deg(Q), \\ \infty & \text{if } \deg(P) > \deg(Q) \end{cases}$$

In the remaining case $\deg(P) = \deg(Q)$ set

$$F(\infty) := \frac{a}{b}$$

where a is the top coefficient of P and b is the top coefficient of Q.

Finally, prove that the map of sets $F\colon S^2 \to S^2$ so defined is a morphism of Riemann surfaces.

Exercise 7. For each quadruple $a, b, c, d \in \mathbf{C}$ such that $ad - bc \neq 0$ show that the rational function $f(z) = \frac{az+b}{cz+d}$ gives rise to an automorphism of Riemann surfaces $S^2 \xrightarrow{\sim} S^2$. (Hint: Try to guess a rational function that gives the inverse morphism.)

Exercise 8. Fill in details of the discussion of projective spaces. Namely, let V be a $\mathbf{C}$-vector space of dimension $n+1 \geqslant 1$. In the followig you are free to assume that $n = 1$, but the case of arbitrary n is not harder.

Recall that we defined $\mathbf{P}(V)$ to be the set of 1-dimensional $\mathbf{C}$-vector subspaces of V equipped with the quotient topology via the surjection

$$V \smallsetminus \{0\} \longrightarrow\!\!\!\!\!\rightarrow \mathbf{P}(V), \quad v \mapsto \mathbf{C}v.$$

Next, for each hyperplane $H \subset V$, i.e. a $\mathbf{C}$-vector subspace of codimension 1, we set

$$\mathbf{P}(V \smallsetminus H) := \{L \in \mathbf{P}(V) \mid L \not\subset H\}.$$

(a) Show that each $\mathbf{P}(V \smallsetminus H)$ is open in $\mathbf{P}(V)$.

(b) Find a set of $n+1$ hyperplanes H such that $\mathbf{P}(V \smallsetminus H)$ cover $\mathbf{P}(V)$.

Pick a splitting $s\colon V \twoheadrightarrow H$, i.e. a $\mathbf{C}$-linear map such that $s|_H = 1_H$. Pick also an isomorphism $t\colon V/H \xrightarrow{\sim} \mathbf{C}$. Given such a triple (H, s, t) consider a map of sets

$$V \smallsetminus H \longrightarrow H, \quad v \mapsto \frac{s(v)}{t([v])}.$$

(c) Show that this map induces a homeomorphism $\mathbf{P}(V \smallsetminus H) \xrightarrow{\sim} H$.

(d) Prove that the collection of maps $\mathbf{P}(V \smallsetminus H) \xrightarrow{\sim} H$ defined by all possible triples (H, s, t) forms a complex manifold atlas on $\mathbf{P}(V)$. In view of (a) and (b) it remains to show that all the transition maps are biholomorphic.

We have thus defined a complex manifold $\mathbf{P}(V)$ of dimension n.

Exercise 9. Show that the Riemann surfaces S^2 and $\mathbf{P}(\mathbf{C} \oplus \mathbf{C})$ are isomorphic.

Finally, here is an exercise which at this stage is probably hard, but doable:

Exercise 10. Let $F\colon S^2 \to S^2$ be a morphism of Riemann surfaces such that $F(S^2) \neq \{\infty\}$. Prove that F is given by a rational function as in Exercise 6.