

Exercise set 12

As in the lectures we denote by Ab the category of abelian groups and by Ab_X the category of sheaves (of abelian groups) on a topological space X . For each topological space X and each abelian group A we denote by $H^i(X, A)$ the i -th sheaf cohomology group of X with coefficients in the constant sheaf $\underline{A}_X$.

Exercise 1. Consider the Riemann sphere $X := S^2$. For each pair of points $P, Q \in X$ prove that the divisors P and Q are equivalent, i.e. that we have an isomorphism of invertible sheaves

$$\mathcal{O}_X(P) \cong \mathcal{O}_X(Q).$$

Exercise 2. Let X be a compact connected Riemann surface. Suppose that there are distinct points P and Q on X such that the divisors P and Q are equivalent. Deduce that $X \cong S^2$.

Exercise 3. Let D be a divisor of degree d on the Riemann sphere $X := S^2$. Show that

$$\dim H^0(X, \mathcal{O}_X(D)) = \max\{0, d + 1\}, \quad \dim H^1(X, \mathcal{O}_X(D)) = \max\{0, -d - 1\}.$$

Exercise 4. (a) Let X be a compact connected Riemann surface of genus $g = 1$, and pick a point $P \in X$. Calculate the dimensions of the spaces of global sections $H^0(X, \mathcal{O}_X(nP))$ for $n = \{0, 1, 2, 3\}$.

(b) Show that the full linear system of $\mathcal{O}_X(3P)$ defines an embedding of X into $\mathbf{CP}^2$.

Next, let $\Lambda \subset \mathbf{C}$ be a lattice, and let $f: \mathbf{C} \rightarrow \mathbf{C}$ be a meromorphic function that is holomorphic outside Λ and satisfies $f(z) = f(z + \lambda)$ for all $z \in \mathbf{C} \setminus \Lambda$ and $\lambda \in \Lambda$. Deduce the following claims:

(c) Suppose that the poles of f in Λ have order at most 1. Then f is constant.

(d) Up to scaling there is a unique f with poles of order exactly 2 at all points in Λ .

Exercise 5. Let X be a compact connected Riemann surface of genus $g \geq 2$. Let V be the dual of the vector space $H^0(X, \Omega_X)$. Show that the full linear system $\phi: V^* \dashrightarrow H^0(X, \Omega_X)$ is base point free.

Consequently, the linear system ϕ determines a morphism $f: X \rightarrow \mathbf{P}(V)$ called the *canonical morphism*. This morphism is not an embedding in general.