Exercise set 11

As in the lectures we denote by Ab the category of abelian groups and by Ab_X the category of sheaves (of abelian groups) on a topological space X. For each topolgical space X and each abelian group A we denote by $H^i(X, A)$ the i-th sheaf cohomology group of X with coefficients in the constant sheaf $\underline{A}_X$.

Exercise 1. Let X be a Riemann surface, and let $\mathcal{D}_X$ be the presheaf that sends an open subset $U \subset X$ to the abelian group of divisors $\mathrm{Div}(U)$.

(a) Prove that $\mathcal{D}_X$ is a sheaf.

(b) Show that there is an isomorphism of sheaves $\mathcal{D}_X \cong \bigoplus_{x \in X} x_* \mathbf{Z}$.

Exercise 2. Let X be a Riemann surface. Construct a bijection between the set of meromorphic functions on X and the set of morphisms of Riemann surfaces $f \colon X \to S^2$ such that $f(X) \neq \{\infty\}$.

Exercise 3. Consider the Riemann sphere $X := S^2$, and pick a point $P \in X$. Show that

$$\Omega_X \cong \mathcal{O}_X(-2P).$$

Hint: you may take $P = \infty$ and work out an explicit isomorphism of invertible sheaves directly from the definitions of Ω_X and $\mathcal{O}_X(-2P)$ using the standard atlas on $X = S^2$.

The last exercise may be somewhat hard at this stage:

Exercise 4. Recall that in Exercise 9 of Set 1 we constructed an isomorphism of Riemann surfaces $S^2 \xrightarrow{\sim} \mathbf{P}(\mathbf{C} \oplus \mathbf{C})$. By Exercise 4 of Set 10 we have the invertible sheaves $\mathcal{O}(n)$, $n \in \mathbf{Z}$, on $\mathbf{P}(\mathbf{C} \oplus \mathbf{C})$, and consequently, on the Riemann sphere S^2. Pick a point $P \in S^2$ and prove that for all n we have an isomorphism

$$\mathcal{O}(n) \;\cong\; \mathcal{O}_{S^2}(nP).$$