

Exercise set 10

As in the lectures we denote by Ab the category of abelian groups and by Ab_X the category of sheaves (of abelian groups) on a topological space X . For each topological space X and each abelian group A we denote by $H^i(X, A)$ the i -th sheaf cohomology group of X with coefficients in the constant sheaf $\underline{A}_X$.

Exercise 1. Let $U \subset \mathbf{C}$ be the open unit disk centered at 0, let $z: U \hookrightarrow \mathbf{C}$ be the holomorphic function which is the identity on U , and let $n \geq 1$ be an integer. Consider the morphism of sheaves $\mu: \mathcal{O}_U \rightarrow \mathcal{O}_U$ given by the formula $f \mapsto fz^n$.

(a) Show that μ is a morphism of $\mathcal{O}_U$ -modules, and that μ is a monomorphism.

Next, for each point $u \in U$ let $\iota_u: \{\cdot\} \hookrightarrow U$ be the corresponding inclusion map.

(b) Show that the morphism $\iota_u^*(\mu)$ is an isomorphism whenever $u \neq 0$.

(c) Show that for $u = 0$ the morphism $\iota_0^*(\mu)$ is zero, and conclude that pullback functors are not exact in general.

(d) Set $\mathcal{M} := \text{coker}(\mu)$; this is naturally an $\mathcal{O}_U$ -module. Show that $\mathcal{M}|_{U \setminus \{0\}} = 0$ and that $\dim_{\mathbf{C}}(\mathcal{M}(U)) = n$.

(e) In the special case $n = 1$ prove that the $\mathcal{O}_U$ -module $\mathcal{M}$ is isomorphic to $\iota_{0*}(\underline{\mathbf{C}}_{\{0\}})$.

(f) (extra) If $n \geq 2$, is it true that $\mathcal{M} \cong \iota_{0*}(\mathcal{N})$ for some sheaf of modules $\mathcal{N}$ on the singleton manifold $\{\cdot\}$? Prove your answer.

Next, let us do some exercises with linear systems. Exercise 3 is especially important! In the following let V be a fixed finite-dimensional $\mathbf{C}$ -vector space. Let also X be a complex manifold. Recall that a *linear system* on X is a pair $(\mathcal{L}, \phi)$ where $\mathcal{L}$ is an invertible $\mathcal{O}_X$ -module and $\phi: V^* \rightarrow H^0(X, \mathcal{L})$ is a morphism of $\mathbf{C}$ -vector spaces. By a slight abuse of terminology we will generally refer to ϕ itself as a linear system.

In the lectures, for every morphism of complex manifolds $g: Y \rightarrow X$ and every linear system ϕ on X we defined the *pullback* linear system $g^*(\phi)$ on Y to be the composite

$$V^* \xrightarrow{\phi} H^0(X, \mathcal{L}) \xrightarrow{\varepsilon_{\mathcal{L}}} H^0(Y, g^*\mathcal{L})$$

where the arrow $\varepsilon_{\mathcal{L}}$ comes from the unit of adjunction $\mathcal{L} \rightarrow g_*g^*\mathcal{L}$ by applying the global sections functor.

Applying this construction to inclusions of points $\{\cdot\} \rightarrow X$ we get for every $x \in X$ a morphism of $\mathbf{C}$ -vector spaces $\phi|_x: V^* \rightarrow \mathcal{L}|_x$. Now recall that ϕ is *base point free* if $\phi|_x \neq 0$ for all $x \in X$. To each base point free linear system ϕ we associate a map of sets $f_{\phi}: X \rightarrow \mathbf{P}(V)$ defined as follows. The vector space $\ker(\phi|_x)$ is by assumption a hyperplane

in V^* . Via the evaluation pairing $V \times V^* \rightarrow \mathbf{C}$, $(v, \ell) \mapsto \ell(v)$ this hyperplane is orthogonal to a line $L_x \subset V$. We then set

$$f_\phi(x) := L_x.$$

Later in the lectures we will see that $f_\phi: X \rightarrow \mathbf{P}(V)$ is a morphism of complex manifolds.

Exercise 2. Let $g: Y \rightarrow X$ be a morphism of complex manifolds, and let $\phi: V^* \rightarrow H^0(X, \mathcal{L})$ be a base point free linear system on X . Show that the induced linear system $g^*(\phi): V^* \rightarrow H^0(Y, g^*\mathcal{L})$ on Y is base point free, and that

$$f_{g^*(\phi)} = f_\phi \circ g.$$

Hint: Use the fact that pullbacks of $\mathcal{O}_X$ -modules are compatible with compositions.

Next, let us introduce an important notion. Let $\mathcal{M}$ be an $\mathcal{O}_X$ -module. For every point $x \in X$ the unit of adjunction $\mathcal{M} \rightarrow \iota_{x*}\iota_x^*\mathcal{M}$ determines a morphism $\mathcal{M}(X) \rightarrow \mathcal{M}|_x$ (by taking the global sections). We say that a point $x \in X$ is a *zero* of a section $s \in \mathcal{M}(X)$ if the image of s in the fiber $\mathcal{M}|_x$ is zero. We denote by $Z(s) \subset X$ the subset of zeroes of s .

Exercise 3. Let $\phi: V^* \rightarrow H^0(X, \mathcal{L})$ be a base point free linear system defining a map $f_\phi: X \rightarrow \mathbf{P}(V)$. Pick a nonzero linear form $\ell \in V^*$ and let $H \subset V$ be the hyperplane that is the kernel of ℓ . Show that

$$f_\phi^{-1}(\mathbf{P}(H)) = Z(\phi(\ell)).$$

In the special case when f_ϕ is the inclusion of a subset this formula means that $Z(\phi(\ell))$ is the intersection of X with $\mathbf{P}(H)$, a so called *hyperplane section* of X .

Exercise 4. In this exercise we assume that $\dim_{\mathbf{C}} V \geq 2$. Let $q: V \setminus \{0\} \rightarrow \mathbf{P}(V)$ be the quotient map. Recall that for each integer n we defined a sheaf $\mathcal{O}(n)$ on $\mathbf{P}(V)$ by the formula

$$\mathcal{O}(n): U \mapsto \{f: q^{-1}(U) \rightarrow \mathbf{C} \text{ is holomorphic and homogeneous of degree } n\}.$$

The homogeneity condition means that for all $z \in q^{-1}(U)$ and $\lambda \in \mathbf{C}^\times$ we have

$$f(\lambda z) = \lambda^n f(z).$$

By construction $\mathcal{O}(0) = \mathcal{O}_{\mathbf{P}(V)}$ is the structure sheaf.

- (a) Show that multiplication by holomorphic functions that are homogeneous of degree 0 provides $\mathcal{O}(n)$ with a structure of an $\mathcal{O}_{\mathbf{P}(V)}$ -module.
- (b) Show that for each n the $\mathcal{O}_{\mathbf{P}(V)}$ -module $\mathcal{O}(n)$ is invertible.
- (c) (extra) For each pair of integers n and n' construct an $\mathcal{O}_{\mathbf{P}(V)}$ -module isomorphism

$$\mathcal{O}(n) \otimes_{\mathcal{O}_{\mathbf{P}(V)}} \mathcal{O}(n') \cong \mathcal{O}(n + n').$$

Elements of V^* are linear forms, and, restricted to $V \setminus \{0\}$, become holomorphic functions that are homogeneous of degree 1. We thus have a map

$$\phi: V^* \hookrightarrow H^0(\mathbf{P}(V), \mathcal{O}(1)).$$

One can in fact show that ϕ is an isomorphism, but we do not need this.

- (d) Show that ϕ is a base point free linear system.
- (e) Show that the induced map $f_\phi: \mathbf{P}(V) \rightarrow \mathbf{P}(V)$ is the identity map.

Next, let us return to de Rham cohomology. Here are some exercises suggested by Ilaria:

Exercise 5. (a) Use the Mayer–Vietoris sequence for de Rham cohomology to prove that

$$H_{\text{dR}}^1(S^n) = 0 \text{ for } n \geq 2$$

and

$$H_{\text{dR}}^k(S^n) \cong H_{\text{dR}}^{k-1}(S^{n-1}) \text{ for } n \geq 2, k \geq 2.$$

Hint: consider $U = S^n \setminus \{(0, \dots, 0, -1)\}$ and $V = S^n \setminus \{(0, \dots, 0, 1)\}$. Note that U and V are diffeomorphic to $\mathbf{R}^n$ and $U \cap V$ is homotopy equivalent to S^{n-1} .

- (b) Deduce the topological invariance of dimension: If $m \neq n$, then $\mathbf{R}^n$ is not homeomorphic to $\mathbf{R}^m$.

Exercise 6. (*Künneth formula*) An open cover $\{U_\alpha\}_{\alpha \in \Lambda}$ of a smooth manifold M is *good* if every intersection $\bigcap_{i \in I} U_i$ with arbitrary $I \subset \Lambda$ is either empty or diffeomorphic to $\mathbf{R}^n$. Every smooth manifold M admits a good cover, see Chapter I, §5 of the book *Differential forms in algebraic topology* by Raoul Bott and Loring Tu. In particular, each compact smooth manifold admits a finite good cover.

- (a) Suppose that a manifold M admits a finite good cover. Prove that $H_{\text{dR}}^k(M)$ is finite-dimensional for all k .
- (b) Let M and N be smooth manifolds admitting finite good covers. Show that for any $0 \leq k \leq \dim M + \dim N$, one has

$$H_{\text{dR}}^k(M \times N) \cong \bigoplus_{i=0}^k H_{\text{dR}}^i(M) \otimes_{\mathbf{R}} H_{\text{dR}}^{k-i}(N).$$

- (c) Deduce that

$$H_{\text{dR}}^k(S^n \times S^n) \cong \begin{cases} \mathbf{R} & \text{if } k = 0, 2n \\ \mathbf{R}^2 & \text{if } k = n \\ 0 & \text{otherwise.} \end{cases}$$

- (d) Deduce that for the n -dimensional torus $T^n = S^1 \times \dots \times S^1$ one has

$$H_{\text{dR}}^k(T^n) \cong \mathbf{R}^{\binom{n}{k}}.$$

The last couple of exercises is very optional (and will not appear at the exam!). We will show that for every complex manifold X the cohomology group $H^1(X, \mathcal{O}_X^\times)$ parametrizes the isomorphism classes of invertible $\mathcal{O}_X$ -modules. The proof of this claim splits into two parts, each presented as an exercise.

Exercise 7. Let X be a topological space and let $\mathcal{F}$ be a sheaf of abelian groups on X . An $\mathcal{F}$ -torsor is a pair $(\mathcal{T}, \mu)$ consisting of a sheaf of sets $\mathcal{T}$ on X and a morphism of sheaves of sets $\mu: \mathcal{F} \times \mathcal{T} \rightarrow \mathcal{T}$ such that

- For every open $U \subset X$ the map $\mu_U: \mathcal{F}(U) \times \mathcal{T}(U) \rightarrow \mathcal{T}(U)$ is a left action of $\mathcal{F}(U)$ on the set $\mathcal{T}(U)$.
- For every open $U \subset X$ either the set $\mathcal{T}(U)$ is empty or $\mathcal{F}(U)$ acts on $\mathcal{T}(U)$ simply transitively.
- Every point $x \in X$ has a neighbourhood U such that $\mathcal{T}(U) \neq \emptyset$.

A morphism of $\mathcal{F}$ -torsors is a morphism of sheaves of sets that commutes with the action of $\mathcal{F}$. Torsors thus form a category. The action of $\mathcal{F}$ on itself (by addition) turns $\mathcal{F}$ into an $\mathcal{F}$ -torsor, called the *trivial torsor*.

- Show that every morphism of $\mathcal{F}$ -torsors is an isomorphism.
- Show that a torsor $\mathcal{T}$ is isomorphic to $\mathcal{F}$ if and only if $\mathcal{T}(X) \neq \emptyset$.

Next, consider extensions of the sheaf $\underline{\mathbf{Z}}_X$ by $\mathcal{F}$, as in Exercise 7 of Set 9. We saw that extensions form a category, and that every morphism of extensions is an isomorphism. Now, to each extension $E = (0 \rightarrow \mathcal{F} \rightarrow \mathcal{E} \rightarrow \underline{\mathbf{Z}}_X \rightarrow 0)$ we associate a sheaf of sets

$$\mathcal{T}_E: U \mapsto \{s \in \mathcal{E}(U) \mid \text{the image of } s \text{ in } \underline{\mathbf{Z}}_X(U) \text{ is equal to } 1\}.$$

The group $\mathcal{F}(U)$ acts on $\mathcal{T}_E(U)$ by addition of sections.

- Prove that $\mathcal{T}_E$ is an $\mathcal{F}$ -torsor.
- Check that the torsor $\mathcal{T}_E$ is isomorphic to $\mathcal{F}$ if and only if the extension E is split.
- Show that the functor $E \mapsto \mathcal{T}_E$ from the category of extensions of $\underline{\mathbf{Z}}_X$ by $\mathcal{F}$ to the category of $\mathcal{F}$ -torsors is fully faithful.
- Given an extension E as above show that every point $x \in X$ has an open neighbourhood U such that the extension $E|_U := (0 \rightarrow \mathcal{F}|_U \rightarrow \mathcal{E}|_U \rightarrow \underline{\mathbf{Z}}_U \rightarrow 0)$ is split. Hint: Use the fact that $\mathbf{Z}$ is a projective object in the category of abelian groups.
- Deduce that the functor $E \mapsto \mathcal{T}_E$ is essentially surjective, and conclude that this functor is an equivalence of categories. Hint: Given a torsor $\mathcal{T}$, there is an open cover $\{U_i\}$ of X such that $\mathcal{T}|_{U_i}$ is trivial for every i . Use part (e) to patch split extensions on each U_i into an extension on X .
- Invoking Exercise 7 of Set 9 conclude that the set of isomorphism classes of $\mathcal{F}$ -torsors is in bijection with the cohomology group $H^1(X, \mathcal{F})$.

Exercise 8. Let X be a complex manifold. (You are free to assume that X is a Riemann surface, but this does not really simplify the problem.) To each invertible $\mathcal{O}_X$ -module $\mathcal{L}$ we associate a sheaf of sets $\mathcal{L}^\times$ defined by the formula

$$\mathcal{L}^\times(U) = \{s \in \mathcal{L}(U) \mid Z(s) = \emptyset\}.$$

Multiplication by invertible holomorphic functions provides $\mathcal{L}^\times$ with an action of $\mathcal{O}_X^\times$.

- Prove that $\mathcal{L}^\times$ is an $\mathcal{O}_X^\times$ -torsor, and check that every isomorphism of $\mathcal{O}_X$ -modules $\mathcal{L}_1 \xrightarrow{\sim} \mathcal{L}_2$ induces a morphism of torsors $\mathcal{L}_1^\times \xrightarrow{\sim} \mathcal{L}_2^\times$.
- Show that the functor $\mathcal{L} \mapsto \mathcal{L}^\times$ is an equivalence the following categories:

- The category that has invertible $\mathcal{O}_X$ -modules as objects and their isomorphisms as morphisms.
 - The category of $\mathcal{O}_X^\times$ -torsors.
- (c) Deduce that the set of isomorphism classes of invertible $\mathcal{O}_X$ -modules is in bijection with the cohomology group $H^1(X, \mathcal{O}_X^\times)$.
- (d) (extra) Assuming that X is a compact connected Riemann surface show that there is an exact sequence of the form

$$0 \longrightarrow H^1(X, \mathbf{Z}) \longrightarrow H^1(X, \mathcal{O}_X) \longrightarrow H^1(X, \mathcal{O}_X^\times) \longrightarrow \mathbf{Z} \longrightarrow 0.$$

One can prove that the image of $H^1(X, \mathbf{Z})$ in $H^1(X, \mathcal{O}_X)$ is a lattice, so that

$$H^1(X, \mathcal{O}_X^\times) \cong T \times \mathbf{Z}$$

where T is a topological torus of dimension $g = \dim_{\mathbf{C}} H^1(X, \mathcal{O}_X)$.