
Solution 1

(a) This is obvious.

(b) The Poisson density function is

$$f(y;\eta) = \frac{\eta^y e^{-\eta}}{y!}, \quad y \in \{0,1,2,\ldots\}, \eta > 0,$$

where y represents the number of deaths and $\eta = \mathrm{E}(y)$ equals the given formula. Then if the y_j are treated as independent this gives log likelihood

$$\ell(\beta) = \sum_{j=1}^{n} \{y_j \log \eta_j(\beta) - \eta_j(\beta) - \log y_j!\}.$$

The components of the algorithm are given by

$$X(\beta)_{n\times 4} = \frac{\partial \eta}{\partial \beta^{\mathrm{T}}}, \quad \frac{\partial \eta_j}{\partial \beta} = \begin{pmatrix} T_j t_j^{\beta_1}\left(1+\beta_2 d_j^{\beta_3}\right) \\ T_j\beta_0 \log(t_j) t_j^{\beta_1}\left(1+\beta_2 d_j^{\beta_3}\right) \\ T_j \beta_0 t_j^{\beta_1} d_j^{\beta_3} \\ T_j\beta_0 t_j^{\beta_1}\beta_2 \log(d_j) d_j^{\beta_3} I(d_j>0) \end{pmatrix},$$

where $I(\cdot)$ denotes an indicator function,

$$\frac{\partial \ell}{\partial \eta_j} = \frac{y_j}{\eta_j} - 1, \quad \frac{\partial^2 \ell}{\partial \eta_j^2} = -\frac{y_j}{\eta_j^2}, \quad j = 1,\ldots,n,$$

and so

$$u_j = \frac{y_j}{\eta_j} - 1, \quad w_j = \mathrm{E}\left(-\frac{\partial^2 \ell}{\partial \eta_j^2}\right) = \frac{\mathrm{E}(y_j)}{\eta_j^2} = \frac{1}{\eta_j},$$

are the jth terms of the $n \times 1$ vector $u(\eta)$ and the $n \times n$ diagonal matrix $W(\eta)$.

Note that some of the y_j equal zero, so we cannot set $\eta_j = y_j$ in the first step of the algorithm, but must instead choose some initial values of β, so that all initial η_j are positive.

Solution 2 In the case of the normal linear model, we have

$$\begin{aligned}
r_{d_j} &= \frac{\operatorname{sign}(\tilde\eta_j - \widehat\eta_j)[2\{\ell_j(\tilde\eta_j;\phi) - \ell_j(\widehat\eta_j;\phi)\}]^{1/2}}{(1-h_{jj})^{1/2}}, \\
&= \frac{\operatorname{sign}(\tilde\eta_j - \widehat\eta_j)}{(1-h_{jj})^{1/2}}\left[2\left\{-\frac{1}{2\phi}(y_j - x_j\widetilde\beta)^2 + \frac{1}{2\phi}(y_j - x_j\widehat\beta)^2\right\}\right]^{1/2}, \\
&= \frac{\operatorname{sign}(y_j - x_j\widehat\beta)}{(1-h_{jj})^{1/2}}\left[2\left\{-\frac{1}{2\phi}(y_j - y_j)^2 + \frac{1}{2\phi}(y_j - x_j\widehat\beta)^2\right\}\right]^{1/2}, \\
r_{d_j} &= \frac{\left(y_j - x_j\widehat\beta\right)}{\sqrt{\phi}(1-h_{jj})^{1/2}},
\end{aligned}$$

because for the saturated model $\eta_j = y_j$. Then we recall that $\phi = \sigma^2$, so replacing $\sqrt{\phi}$ by its estimate s we retrieve the standardized linear model residuals.

Similarly, we have

$$\begin{aligned} r_{Pj} &= \frac{u_j(\widehat{\beta})}{\{w_j(\widehat{\beta})(1-h_{jj})\}^{1/2}} \\ &= \frac{(y_j - x_j\widehat{\beta})/\phi}{\{w_j(\widehat{\beta})(1-h_{jj})\}^{1/2}} \end{aligned}$$

and we have $w_j(\widehat{\beta}) = \phi^{-1}$, estimated by $1/s^2$, which leads to

$$r_{Pj} = \frac{(y_j - x_j\widehat{\beta})}{\{s(1-h_{jj})\}^{1/2}}.$$

In both cases, we retrieve the classical standardized residuals of the linear normal model.

As $r_{Pj} = r_{Dj}$, we have

$$r_j^* = r_{Dj} + \frac{1}{r_{Dj}} \log(r_{Pj}/r_{Dj}) = r_{Dj}.$$

Solution 3 The density function is

$$\begin{aligned} f(y;\mu,\nu) &= \frac{1}{\Gamma(\nu)} y^{\nu-1} \left(\frac{\nu}{\mu}\right)^{\nu} \exp\left(-\frac{\nu y}{\mu}\right), \\ &= \exp\left\{-\frac{\nu y}{\mu} + \nu \log\left(\frac{\nu}{\mu}\right) + (\nu-1)\log y - \log\Gamma(\nu)\right\}, \\ &= \exp\left\{\nu\left(-y/\mu - \log\mu\right) + \nu\log\nu + (\nu-1)\log y - \log\Gamma(\nu)\right\}, \end{aligned}$$

and we deduce that

$$\theta = -\frac{1}{\mu}, \quad b(\theta) = -\log(-\theta), \quad \phi = \frac{1}{\nu}, \quad c(y,\phi) = \phi^{-1}\log(\phi) + \left(\phi^{-1} - 1\right)\log y - \log\Gamma(\phi^{-1}),$$

where $\theta < 0$ and $\phi > 0$. For the mean and variance we have

$$\mathrm{E}(Y) = b'(\theta) = -\frac{1}{\theta} = \mu, \quad \mathrm{var}(Y) = \phi b''(\theta) = \frac{1}{\nu\theta^2} = \frac{\mu^2}{\nu},$$

so the variance function is

$$V(\mu) = b''\{b'^{-1}(\mu)\} = \theta^{-2}\Big|_{\theta=-1/\mu} = \mu^2.$$

The canonical link function satisfies $\eta = \theta = -1/\mu$. The log link $\eta = \log\mu$ seems generally preferable, because it ensures that $\mu > 0$.

Solution 4 As Y is binary,

$$\mu = \mathrm{E}(Y) = 1 - \mathrm{P}(Y=0) = 1 - \mathrm{P}(X=0) = 1 - \exp\{-\exp(x^{\mathrm{T}}\beta)\} = 1 - \exp\{-\exp(\eta)\},$$

giving the complementary log-log link function

$$\eta = g(\mu) = \log\{-\log(1-\mu)\}, \quad 0 < \mu < 1.$$