Regression Methods: Problems MATH-408
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Problem 1 Which of the following can be written as linear regression models, (i) as they are, (ii) when a single parameter is held fixed, (iii) after transformation? For those that can be so written, give the response variable and the form of the design matrix. (a) $y = \beta_0 + \beta_1/x + \beta_2/x^2 + \varepsilon$; (b) $y = \beta_0/(1+\beta_1 x)+\varepsilon$; (c) $y = 1/(\beta_0+\beta_1 x+\varepsilon)$; (d) $y = \beta_0+\beta_1 x^{\beta_2}+\varepsilon$; (e) $y = \beta_0+\beta_1 x_1^{\beta_2}+\beta_3 x_2^{\beta_4}+\varepsilon$.

Problem 2 Data are available on the weights of two groups of three rats at the beginning of a fortnight, x, and at its end, y. During the fortnight, one group was fed normally and the other group was fed a growth inhibitor. Consider a linear model for the weights,

$$y_{gj} = \alpha_g + \beta_g x_{gj} + \varepsilon_{gj}, \quad g = 1, 2, \quad j = 1, \ldots, 3.$$

(a) Write down the response and parameter vectors and design matrix for the model above.

(b) The model is to be reparametrized in such a way that it can be specialized to (i) two parallel lines for the two groups, (ii) two lines with the same intercept, (iii) one common line for both groups, just by setting parameters to zero. Give one design matrix which can be made to correspond to (i), (ii), and (iii), just by dropping columns.

Problem 3 In the notation of the course, consider minimising the sum of squares $\|y - X\beta\|^2$.

(a) Check the computation leading to the least squares estimates $\widehat{\beta} = (X^{\mathrm{T}}X)^{-1}X^{\mathrm{T}}y$.

(b) Verify that the hat matrix $H = X(X^{\mathrm{T}}X)^{-1}X^{\mathrm{T}}$ is symmetric and idempotent, and show that its eigenvalues consist of p 1s and $n - p$ 0s.

(c) If $X'_{n\times q}$ is a subset of the columns of $X_{n\times p}$, show that the corresponding hat matrices satisfy

$$H'(H - H') = H'(I - H) = H'(I - H') = H(I - H) = 0.$$

Problem 4 (Generalized and weighted least squares)

(a) If $y \sim (X\beta, \sigma^2 Q)$, where $Q_{n\times n}$ is known, symmetric and positive definite, show that the resulting estimates are

$$\widehat{\beta} = (X^{\mathrm{T}}WX)^{-1}X^{\mathrm{T}}Wy,$$

where $W = Q^{-1}$, and give expressions for the corresponding hat matrix and residual sum of squares.

The $\widehat{\beta}$ are called *weighted least squares (WLS)* estimates when Q is diagonal, and *generalized least squares (GLS)* estimates when Q has non-zero off-diagonal elements.

(b) By considering the expression for $\widehat{\beta}$ when Q is diagonal, explain the term *weighted least squares (WLS)* estimates.