GRAPH THEORY
I. TOMON

Final Exam

Spring 2019
July 4, 2019

Last name :
Sciper :

First name :
Section :

Exercise :	1	2	3	4	5	6	7	8	9	10	Σ	
Score :												

- You may not use a calculator on this exam.
- No additional materials are permitted.
- Even if you cannot solve a problem, write down your ideas.

- Each question is worth 10 points.
- All graphs are simple, and have at least one vertex.
- K_n denotes the complete n-vertex graph.
- $K_{a,b}$ denotes the complete bipartite graph with parts of size a and b.

Time : 16.15 – 19.15

Carefully read the small print at the bottom of the page. The problems are in no particular order.

1. Prove that in any connected graph G, there is a walk that contains every edge of G exactly twice.

2. Prove Dirac's theorem: If a graph G on $n \geq 3$ vertices has minimum degree at least $\frac{n}{2}$, then it contains a Hamilton cycle.

3. Prove that every connected planar graph on $n \geq 3$ vertices has a triangular face or a vertex of degree at most 3. (A triangular face is one whose boundary has length 3).

4. Let G be a bipartite graph with parts of size $2n$ and minimum degree at least n. Prove that G has a perfect matching.

5. Prove that if a graph G on n vertices does not contain $K_{2,2}$ as a subgraph, then G has at most $n^{3/2}$ edges.

6. Let $n \geq 2$ be an integer, and $R(n, n)$ be the corresponding Ramsey number. Show that any sequence of $N \geq R(n, n)$ distinct numbers $a_1, \ldots, a_N$ contains a monotone (increasing or decreasing) subsequence of length n.

7. Let G be a bipartite graph. Prove that if λ is an eigenvalue of the adjacency matrix of G, then $-\lambda$ is also an eigenvalue.

8. Let G be a graph, and suppose $d \geq 0$ is the smallest number such that G is d-degenerate. Prove that G has at least $\frac{d(d+1)}{2}$ edges.
(A graph is d-degenerate if each of its subgraphs has a vertex of degree at most d.)

9. Prove the fan lemma: Let k be a positive integer. If G is a k-connected graph, then for every vertex s, and for every set T of at least k vertices, there are k paths from s to T in G that are vertex-disjoint except for their starting vertex s.

10. Let n and k be positive integers. Show that the edges of K_n can be colored with k colors so that the number of monochromatic triangles is at most $\frac{1}{k^2}\binom{n}{3}$.
(A monochromatic triangle is a 3-cycle whose edges have the same color.)

You may **not** use any results from the lecture notes or problem sets, with the following exceptions. When you use a result, it should be clearly indicated.
1: You may cite any fact from the lecture notes.
3: You may cite any fact from the lecture notes.
4: You may cite any fact from the lecture notes.
6: You may cite any fact from the lecture notes.
7: You may cite any fact from the lecture notes.
8: You may cite any fact from the lecture notes.
9: You may cite Menger's theorem.
10: You may cite any fact from the lecture notes.