Graph Theory
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Spring 2018
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Final Exam

Last name :

First name :

Sciper :

Section :

Exercise :	1	2	3	4	5	6	7	8	9	Σ	B	
Score :												

- You may not use a calculator on this exam.
- No additional materials are permitted.
- Even if you cannot solve a problem, write down your ideas.

- Each question is worth 10 points.
- All graphs are simple, and have at least one vertex.
- $\chi(G)$ stands for the chromatic number and $R(s, t)$ stands for the Ramsey number.

Time : 08.15 – 11.15

Carefully read the small print at the bottom of the page. The problems are in no particular order.

1. Prove that if G is a connected planar graph on n vertices that has finite girth g, then it has at most $\frac{g}{g-2}(n-2)$ edges.

2. Show that in any tree containing an even number of edges, there is at least one vertex with even degree.

3. Prove that a K_3-free graph on n vertices contains at most $\lfloor \frac{n^2}{4} \rfloor$ edges.

4. Let G be a connected graph with maximum degree Δ, such that $\chi(G) = \Delta + 1$. Prove that G is Δ-regular.

5. (a) [**7 points**] Show that if for some real number $0 \le p \le 1$ we have $\binom{n}{s} p^{\binom{s}{2}} + \binom{n}{t} (1-p)^{\binom{t}{2}} < 1$, then $R(s,t) > n$.

   (b) [**3 points**] Deduce that there is a positive constant c such that $R(4,t) \ge c \cdot \frac{t^{3/2}}{\log^{3/2} t}$ for every integer $t \ge 2$.

6. Prove that a connected graph has an Eulerian tour if and only if each vertex has even degree.

7. Let A be an $n \times m$ matrix of non-negative real numbers such that the sum of the entries is an integer in every row and in every column. Prove that there is an $n \times m$ matrix B of non-negative integers such that in every row and in every column, the sum of the entries in B is the same as in A.

8. Describe an efficient algorithm for finding a minimum-weight spanning tree in a connected weighted undirected graph, and prove that it indeed returns such a tree.

9. Let G be a k-connected graph with at least $2k$ vertices for some $k \ge 2$.

   (a) [**5 points**] Prove that G contains a cycle of length at least k.

   (b) [**5 points**] Prove that G contains a cycle of length at least $2k$.

You may not use any results from the lecture notes or problem sets, with the following exceptions. When you use a result, it should be clearly indicated.

1: You may use any fact from the lecture notes.

2: You may use any fact from the lecture notes or problem sets.

4: You may use any fact from the lecture notes or problem sets.

5: You may use the facts $1 - x \le e^{-x}$ for $x > 0$, and $\binom{a}{b} \le \frac{a^b}{2}$ for $a > b > 1$ integers.

7: You may use any fact from the lecture notes.

9: You may use any fact from the lecture notes or problem sets.