Final exam – Solutions

Graph Theory 2017 – EPFL – Andrey Kupavskii

1. *Show that if $G = (A \cup B, E)$ is a bipartite graph such that $|N(S)| \geq |S| - d$ holds for some integer $d \geq 0$ and every $S \subseteq A$, then G has a matching with at least $|A| - d$ edges.*

   Solution. Add d new vertices to B, each connected to all vertices in A; let G' be the new graph. Then G' has $|N_{G'}(S)| \geq |S|$ for every $S \subset A$ (S has at least $|S| - d$ neighbors from G, and is connected to the d new vertices). By Hall's Theorem, G' has a matching for A, which has $|A|$ edges. At most d of these edges contain a new vertex of G', which leaves at least $|A| - d$ edges from G.

2. *The edges of K_{11} are colored red or blue such that every edge gets exactly one color. Show that the graph of the red edges and the graph of the blue edges cannot both be planar.*

   Solution. We know that any planar graph on 11 vertices has at most $3 \cdot 11 - 6 = 27$ edges. Since K_{11} has $\binom{11}{2} = 55$ edges, then either red or blue graph will have at least 28 edges, and thus cannot be planar.

3. *Prove that for $t > 3$ the Ramsey number of K_t satisfies $R(K_t, K_t) \geq 2^{t/2}$.*

   Solution. Color the edges of K_n randomly and independently of each other into red and blue (both possibilities have probability 1/2). Then the probability that on a given set of t vertices we got either a red or a blue clique is $2^{1-\binom{t}{2}}$. Therefore, by the linearity of expectation, the expected number of monochromatic cliques is

$$X := \binom{n}{t} 2^{1-\binom{t}{2}}.$$

   There exists a graph G which has at most X monochromatic cliques.

   If we have $X < 1$, then, since the number of monochromatic cliques is integer, it must be equal to 0 in G.

   We wish to choose $n = n(t)$, so that $2^{1-\binom{t}{2}}\binom{n}{t} < 1$. Let us make some estimates:

$$\binom{n}{t} < \frac{n^t}{t!} \leq \frac{n^t}{2^{1+t/2}},$$

   where the last inequality holds for $t > 3$. Then

$$2^{1-\binom{t}{2}}\binom{n}{t} < 2^{1-t(t-1)/2-1-t/2} n^t = 2^{-t^2/2} n^t.$$

   This is smaller than 1 if $n < 2^{t/2}$, which implies that $R(K_t, K_t) \geq 2^{t/2}$.

4. *Prove that for every $k \geq 2$, there is an integer N such that whenever the numbers $\{1, \ldots, N\}$ are colored with k colors, three are three numbers $1 \leq a, b, c \leq N$ satisfying $ab = c$ that have the same color.*

Solution. We assume that $a, b, c \geq 2$ (otherwise, $a = b = c = 1$ is a trivial solution). According to Schur's theorem, there is a K such that every coloring of $[K]$ with k colors contains three numbers x, y, z satisfying $x+y = z$. Now let $N = 2^K$ and take an arbitrary k-coloring c of $[N]$. Let d be a coloring of $[K]$ defined by $d(i) = c(2^i)$. By Schur's theorem, there are x, y, z such that $x + y = z$ and $d(x) = d(y) = d(z)$. But then $2^x \cdot 2^y = 2^{x+y} = 2^z$ and $d(x) = c(2^x) = c(2^y) = c(2^z)$, which is what we wanted.

5. *State the Gale-Shapley algorithm and prove that it outputs a stable matching.*

Consider a bipartite graph G with parts A, B, where $|A| = |B| = n$, and in which each vertex has a (strict) order of preferences for all the vertices of the other part. We say that a perfect matching is *stable*, if there is no pair $a \in A, b \in B$, such that both of them would prefer the other to the vertex they are currently matched to.

Below we present an algorithm of Gale and Shapley, which allows to construct such a stable matching.

The Gale-Shapley Algorithm to find a stable matching M in a complete bipartite graph G with bipartition $V(G) = A \cup B$, $|A| = |B|$

(1) Set $M = \emptyset$;

(2) Iterate:

   (a) Take an unmatched vertex $a \in A$ and let $b \in B$ be the vertex that a prefers among the ones a has not tried yet.

   (b) a "proposes" to b: If b is unmatched or b is matched to a', but prefers a over a', then "accept" a and "reject" a': put $M := M - a'b + ab$. Otherwise, "reject": leave M unchanged;

   (c) If there is no more unmatched vertices in A that have someone left on the list, then go to (3);

(3) Return M.

Proposition 1. *The matching M that the algorithm outputs is stable.*

Proof. First we show that M is perfect. Indeed, if there is a pair of vertices $a \in A, b \in B$, such that both are not in the matching, then a must have proposed to b at some point. However, if a vertex $b \in B$ is in M at some step of the algorithm, then it stays in M.

Next, we show that the matching is stable. Assume that $ab \notin M$. Upon completion of the algorithm, it is not possible for both a and b to prefer each other over their current match. If a prefers b to its match, then a must have proposed to b before its current match. If b accepted its proposal, but is matched to another vertex at the end, then b prefers the current match of b over a. If b rejected the proposal of a, then b was already matched to a vertex that is better for b. $\square$

6. *Let G be a connected graph having an even number of edges such that all the degrees are even. Prove that the edges of G can be colored by red and blue in such a way that every vertex has the same number of red and blue edges touching it.*

Solution. Since the graph is connected, and all the degrees are even, the graph contains an Euler tour. Since the number of edges in the graph is even, the length of the Euler tour, being equal to the total number of edges, is even, and so we can color the edges along the tour in red and blue so that the colors alternate. Then if a vertex v has degree $2d$, then it is visited d times by the Euler tour, and each such visit involves two edges of different colors. Hence every vertex has the same number of red and blue edges.

7. *Let G be a k-connected graph for some $k \geq 2$. Show that for any k vertices in G, there is a cycle in G that passes through all of them.*

   Solution. We use (without proof) the following proposition, which follows from Menger's theorem and was shown on the lectures:

   Proposition 2. *Let G be a k-connected graph. For every $x \in V(G)$ and $U \subset V(G)$ with $|U| \geq k$, there are k paths from x to U that are disjoint aside from x, with each path having exactly one vertex from U.*

   We use induction on k. The case $k = 2$ follows directly from the case $k = 2$ of Menger's Theorem: two internally vertex disjoint paths between two vertices give a cycle.

   Assume $k > 2$ and pick any $x \in K$. By induction, G has a cycle C containing $K \backslash \{x\}$. If $x \in V(C)$, we are done, so we can assume that $x \notin V(C)$.

   Suppose that $|V(C)| = k-1$. By Proposition **??** and the fact that G is $(k-1)$-connected, there are $k-1$ paths from x to C that are disjoint aside from x, each containing exactly one vertex of C. We can use any two of the paths from x that end at adjacent vertices $y, z \in V(C)$ to obtain a cycle containing x as well as $K \backslash \{x\}$: Remove the edge yz from C, and replace it by the path that goes from y to x and then from x to z. Since these paths were disjoint aside from x, and also contain no other vertices from C, this indeed gives a cycle.

8. *Let G be a graph on $n \geq 3$ vertices with at least $\alpha(G)$ vertices of degree $n-1$. Show that G contains a Hamilton cycle.*

   Solution. There are several solutions to the problem. A direct proof using maximal cycles and then rotating and prolonging the cycle was given in the exercise sheets. A simple way is as follows. On the lectures we proved that if $\kappa(G) \geq \alpha(G)$, then the graph has a Hamilton cycle. We just need to verify the condition $\kappa(G) \geq \alpha(G)$. Clearly, the graph contains more than $\alpha(G)$ vertices. Moreover, deleting at most $\alpha(G) - 1$ vertices doesn't disconnect the graph, since every pair of vertices is connected via (at least) one vertex of degree $n-1$, which was not deleted. Therefore, $\kappa(G) \geq \alpha(G)$.

9. (a) *Let G and H be two graphs on the same vertex set. Prove that $\chi(G \cup H) \leq \chi(G)\chi(H)$.*

   (b) *Let $k \geq 1$ and $n \geq 2^k + 1$ be integers, and suppose that $K_n = G_1 \cup \cdots \cup G_k$ for some graphs $G_1, \ldots, G_k$. Prove that for some $i \in \{1, \ldots, k\}$, G_i is not bipartite.*

   Solution. Denote the common vertex set of the graphs by V.

   (a) Consider the proper colorings $c_1 : V \to \{1, \ldots, \chi(G)\}, c_2 : V \to \{1, \ldots, \chi(H)\}$ of G and H, respectively. Color $v \in V$ into the color $(c_1(v), c_2(v))$. Then, if the two vertices receive the same color, then they have the same color in both c_1 and c_2, which means that there is no edge between them neither in G nor in H. Therefore, it is a proper coloring.

   (b) By induction from (a), if $K_n = G_1 \cup \ldots \cup G_k$, then $\chi(K_n) \leq \chi(G_1 \cup \ldots \cup G_{k-1})\chi(G_k) \leq \ldots \leq \chi(G_1)\chi(G_2)\ldots\chi(G_k)$. Since $\chi(K_n) = n = 2^k + 1 > 2^k$, we have that for at least one $j = 1 \ldots, k$ $\chi(G_j) > 2$.