Graph Theory
A. Kupavskii

Final Exam

Spring semester 2017
July 4, 2017

Last name :
First name :
Sciper :
Section :

Exercise :	1	2	3	4	5	6	7	8	9	Σ
Score :										

- You may not use a calculator on this exam.
- No additional materials are permitted.
- Even if you cannot solve a problem, write down your ideas.

- All questions have the same value.
- All graphs are simple.
- $\alpha(G)$ stands for the independence number and $\chi(G)$ stands for the chromatic number.

Time : 08.15 – 11.15

Carefully read the small print at the bottom of the page. The problems are in no particular order.

1. Show that if $G = (A \cup B, E)$ is a bipartite graph such that $|N(S)| \geq |S| - d$ holds for some integer $d \geq 0$ and every $S \subseteq A$, then G has a matching with at least $|A| - d$ edges.
2. The edges of K_{11} are colored red or blue such that every edge gets exactly one color. Show that the graph of the red edges and the graph of the blue edges cannot both be planar.
3. Prove that for $t > 3$ the Ramsey number of K_t satisfies $R(K_t, K_t) \geq 2^{t/2}$.
4. Prove that for every $k \geq 2$, there is an integer N such that whenever the numbers $\{1, \ldots, N\}$ are colored with k colors, three are three numbers $1 \leq a, b, c \leq N$ satisfying $ab = c$ that have the same color.
5. State the Gale-Shapley algorithm and prove that it outputs a stable matching.
6. Let G be a connected graph having an even number of edges such that all the degrees are even. Prove that the edges of G can be colored by red and blue in such a way that every vertex has the same number of red and blue edges touching it.
7. Let G be a k-connected graph for some $k \geq 2$. Show that for any k vertices in G, there is a cycle in G that passes through all of them.
8. Let G be a graph on $n \geq 3$ vertices with at least $\alpha(G)$ vertices of degree $n-1$. Show that G contains a Hamilton cycle.
9. (a) Let G and H be two graphs on the same vertex set. Prove that $\chi(G \cup H) \leq \chi(G)\chi(H)$.
   (b) Let $k \geq 1$ and $n \geq 2^k + 1$ be integers, and suppose that $K_n = G_1 \cup \cdots \cup G_k$ for some graphs $G_1, \ldots, G_k$. Prove that for some $i \in \{1, \ldots, k\}$, G_i is not bipartite.

You may not use any results from the lecture notes or problem sets, with the following exceptions. When you use a result, it should be clearly indicated.

1: You may use any fact from the lecture notes.
2: You may use any fact from the lecture notes or problem sets.
3: You may use the fact that $t! \geq 2^{1+t/2}$ if $t > 3$.
4: You may use any fact from the lecture notes.
6: You may use any fact from the lecture notes or problem sets.
7: This is a theorem from the lectures. You may use any previously established results.
8: You may use any fact from the lecture notes.
9: You may use any fact from the lecture notes or problem sets.