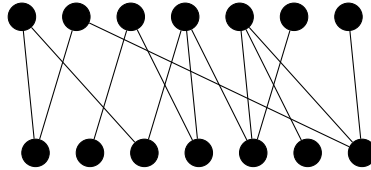


Graph Theory - Problem Set 8 (Solutions)

November 7, 2024

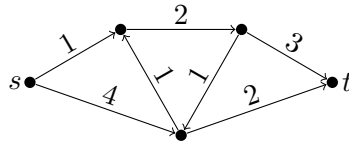
Exercises

- Find a minimum vertex cover in the following graph.

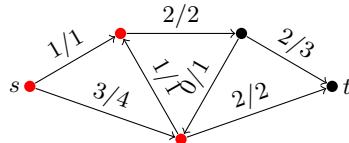


Solution. Since this graph has a perfect matching, the vertices from the top part form a minimum vertex cover.

- Find a maximum flow from s to t and a minimum s - t cut in the following network.



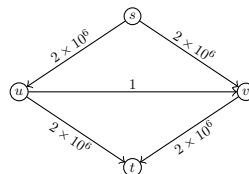
Solution. The only maximum flow here (with value 4) is the following:



An example of a minimum cut (of capacity 4) is shown in red vs black.

- Construct a network on four vertices for which the Ford-Fulkerson algorithm may need more than a million iterations, depending on the choice of augmenting paths.

Solution. Consider the following network:



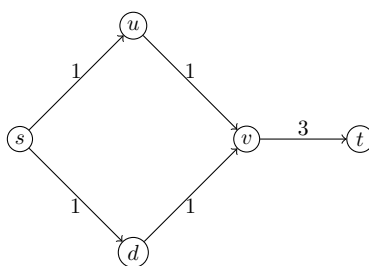
Clearly, the maximum value of a flow is 2 million, but the Ford-Fulkerson algorithm might always choose a path that uses the edge uv by alternately choosing $suvt$ and $svut$. This way the value of the flow increases by 1 in every step, so it takes 2 million improvements to reach a maximum flow.

4. Let G be a network with source s , sink t , and integer capacities. Prove or disprove the following statements:

- (a) If all capacities are even then there is a maximal flow f such that $f(e)$ is even for all edges e .
- (b) If all capacities are odd then there is a maximal flow f such that $f(e)$ is odd for all edges e .

Solution.

- (a) Dividing all capacities by two, we obtain a network with integral capacities. Therefore, it has an integral maximum flow. Multiplying this flow by 2, we get an even maximum flow on the original network.
- (b) Counterexample:



Problems

5. Let G be a graph on n vertices

- (a) Show that $\tau(G) + \alpha(G) = n$.
- (b) Show that if G is bipartite then $\nu(G) + \alpha(G) = n$.

Solution.

- (a) First note that $X \subseteq V(G)$ is a vertex cover if and only if $V(G) \setminus X$ is an independent vertex set. To prove the case $\Rightarrow$, suppose by the contradiction that there exist vertices $u, v \in V(G) \setminus X$ such that $uv \in E(G)$. Then the edge uv is not covered by any of the vertices in X , which is in contradiction with the assumption that X is a vertex cover. One can prove the $\Leftarrow$ case by a similar reasoning.

Now take the vertex cover X with $|X| = \tau(G)$. We know $V(G) \setminus X$ is an independent vertex set, so we have

$$n = |X| + |V(G) \setminus X| = \tau(G) + |V(G) \setminus X| \leq \tau(G) + \alpha(G).$$

On the other hand, consider the independent vertex set Y with $|Y| = \alpha(G)$. By the remark, $V(G) \setminus Y$ forms a vertex cover.

$$n = |Y| + |V(G) \setminus Y| = \alpha(G) + |V(G) \setminus Y| \geq \alpha(G) + \tau(G).$$

Therefore, we have $\tau(G) + \alpha(G) = n$.

(b) It follows from part (a) and the König's Theorem ($\nu(G) = \tau(G)$).

6. Let G be a network with source s , sink t , and integer capacities. Prove that an edge e is saturated (i.e., the flow uses its full capacity) in every maximum s - t flow if and only if decreasing the capacity of e by 1 would decrease the maximum value of an s - t flow in G .

Solution. Let k be the value of the maximum flow, i.e., the capacity of a minimum cut in G . We argue by contradiction for both directions.

$\Rightarrow$: Suppose decreasing the capacity does not decrease the size of the max flow. Then there is a flow in this new network G' of value k that does not use the full capacity of e in G . This corresponds to a flow of size k in G with the same property, contradicting our assumption.

$\Leftarrow$: Suppose e is not saturated in every maximum s - t flow. If we could assume that this flow is an integer flow, then we would be done: This flow would have value k in G' , contradicting the assumption. But we don't quite know that we can assume this, so let's do something else instead:

Indeed, if e is not saturated in some maximum flow, then e does not occur in any min cut (otherwise the size of this flow would be less than the capacity of a min cut). Now we do know that every cut has an integer capacity, so if e is not in a k -cut, then every cut containing e has capacity at least $k + 1$. Hence decreasing the capacity of e by 1 will not create any cut of size k , therefore the size of the maximum flow does not decrease. Contradiction.

7. Deduce Hall's theorem from the max-flow min-cut theorem.

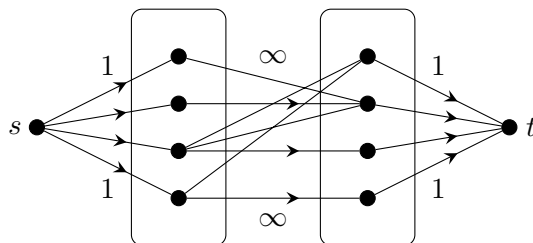
Hint: Consider a network derived from the bipartite graph similar to the proof of König theorem.

Solution. Consider the bipartite graph $G = (A \cup B, E)$ that satisfies Hall's conditions. Let us make a network out of G . Add a source s , connect it to all vertices of A by oriented edges of capacity 1. Analogously, add a sink t and connect all vertices of B to it by edges of capacity 1. Let the edges of G be oriented from A to B and their capacities to be infinite. We stress that all the edges are now oriented in the direction from s to t (see the figure below).

If there is a integer flow of value $|A|$ in G , then the edges (x, y) satisfying $x \in A, y \in B$, and $f(x, y) = 1$ constitute a matching of A in G , and we are done. Otherwise, there is a cut (X, Y) of capacity $k < |A|$. We know that

$$|A \cap Y| + |B \cap X| = k < |A| = |A \cap X| + |A \cap Y|,$$

from which we conclude that $|B \cap X| < |A \cap X|$. Let $W = A \cap X$. The set $N(W)$ is contained in $B \cap X$, as otherwise there would be an infinite-capacity edge crossing from X to Y . Thus, $|N(W)| = |B \cap X| < |W|$, and we verified that when a perfect matching does not exist, there is a set W violating Hall's criterion.



8. Let A be an $n \times m$ matrix of non-negative real numbers such that the sum of the entries is an integer in every row and in every column. Prove that there is an $n \times m$ matrix B of non-negative integers with the same sums as in A , in every row and every column.

Solution. Let $a_1, \dots, a_n$ and $b_1, \dots, b_m$ be the sums of the entries in the rows and columns of A , respectively. Consider the complete bipartite graph $G = (P \cup Q, E)$, where $P = \{p_1, \dots, p_n\}$ and $Q = \{q_1, \dots, q_m\}$. Let us make a directed network out of G . Orient all edges from P to Q , and set their capacity to infinity. Add a source s , connect it to all vertices of P by oriented edges of capacity a_i for the vertex p_i . Analogously, add a sink t and connect all vertices of Q to it by oriented edges of capacity b_j for the vertex q_j .

Clearly, the minimum capacity of a cut is $a_1 + \dots + a_n = b_1 + \dots + b_m$. Since all capacities are integral, the Ford-Fulkerson theorem shows that there is an integer flow with maximum value. We can then define the entry B_{ij} of the matrix B to be the value of this flow on the edge $p_i q_j$.