Graph Theory - Problem Set 7 (Solutions)

October 31, 2024

Exercises

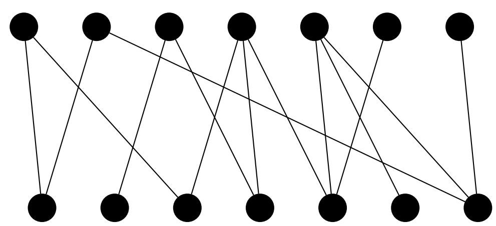
1. Construct preference lists for the vertices of $K_{3,3}$ so that there are multiple stable matchings.

   Solution. For example, for parts $\{1, 2, 3\}$ and $\{x, y, z\}$, take

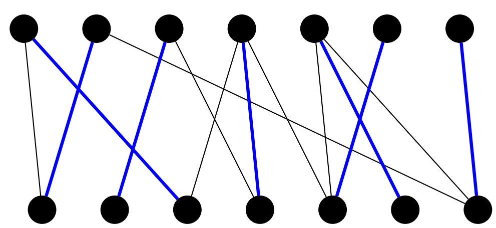
$$\begin{pmatrix} 1: & z & y & x \\ 2: & y & x & z \\ 3: & x & z & y \end{pmatrix} \qquad \begin{pmatrix} x: & 2 & 1 & 3 \\ y: & 3 & 1 & 2 \\ z: & 1 & 2 & 3 \end{pmatrix}$$

   Here $\{(1, z), (2, y), (3, x)\}$ is a stable matching (optimal for the numbers), and another one is $\{(1, z), (2, x), (3, y)\}$ (optimal for the letters).

2. Find a maximum matching in the following graph.



   Solution. It has a perfect matching, for example:



3. Construct a 2-regular graph without a perfect matching.

   Solution. Take an odd cycle.

4. Let G be a bipartite graph on $2n$ vertices such that $\alpha(G) = n$.

   (a) Show that both parts of G contain n vertices.

   (b) Check that Hall's condition holds for G and then deduce that G has a perfect matching.

   Solution. Let $V(G) = A \cup B$ be the bipartition of G.

   (a) We have $|A|, |B| \leq \alpha(G)$, which implies that $2n = |A| + |B| \leq 2\alpha(G) = 2n$. Therefore, we get $|A| = |B| = \alpha(G) = n$.

(b) For $S \subseteq A$, if $|N(S)| < |S|$, then $S \cup (B \setminus N(S))$ is an independent set of size

$$|S \cup (B \setminus N(S))\,| = |S| + |B| - N(S) > |B| = n,$$

a contradiction to the assumption $\alpha(G) = n$. By Hall's theorem, there is a matching covering A. Since $|A| = |B|$, this is indeed a perfect matching.

Problems

5. Prove the following "defect" version of Hall's theorem:

   If $G = (A \cup B, E)$ is a bipartite graph such that $|N(S)| \geq |S| - d$ holds for every $S \subseteq A$, then G has a matching with at least $|A| - d$ edges.

   Solution. Add d new vertices to B, where each is connected to all vertices in A. Denote the new graph as G'. Then G' has $|N_{G'}(S)| \geq |S|$ for every $S \subset A$, since S has at least $|S| - d$ neighbors from G and d neighbors being the new vertices. By Hall's theorem, G' has a matching covering A, which has $|A|$ edges. At most d edges of this matching contain a new vertex of G', which leaves at least $|A| - d$ edges from G.

6. An $r \times s$ *Latin rectangle* is an $r \times s$ matrix A with entries in $\{1, \ldots, s\}$ such that each integer occurs at most once in each row and at most once in each column. An $s \times s$ Latin rectangle is called a *Latin square*. Prove that every $r \times s$ Latin rectangle can be extended to an $s \times s$ Latin square.

   Hint: Consider a bipartite graph that models the constraints of any $i \in \{1, \ldots, s\}$ appearing only once in each row and column.

   Solution. Define a bipartite graph whose vertex set consists of two copies of $\{1, \ldots, s\}$, call them S_1 and S_2. We connect $i \in S_1$ with $j \in S_2$ if the i-th column of the $r \times s$ Latin rectangle does not contain the number j. What we are looking for is a matching that matches S_1, since then we can put numbers on row $r + 1$ such that no number is repeated in that row, and no number is repeated in a column.

   A column $i \in S_1$ contains r distinct numbers, so there are $s - r$ numbers that it does not contain. That means that the vertex $i \in S_1$ has degree $s - r$. On the other hand, a number $j \in S_2$ occurs exactly once in each of the r rows, and at most once in any of the s columns. Hence there are $s - r$ columns that do not contain j, so the degree of $j \in S_2$ is $s - r$. Therefore the graph is $(s - r)$-regular, so there is a perfect matching.

7. Let G be a bipartite graph with both parts of the same size $2n$ and minimum degree at least n. Prove that G has a perfect matching.

   Solution. Denote the two parts of G as A and B. To show that G has a perfect matching, by Hall's theorem it suffices to check Hall's condition for A. Take any $S \subseteq A$.

   - If S is empty, then $|N(S)| = |S| = 0$, so the condition holds.
   - If $1 \leq |S| \leq n$, then $|N(S)| \geq n \geq |S|$ since any vertex in X has at least n neighbors in B.
   - If $|S| > n$, then $N(S) = B$ since every vertex v in B has at least n neighbors in A, so it must have a neighbor in S (otherwise the disjoint union $S \cup N(v)$ would contain more than $2n$ vertices in A). Therefore $|N(S)| = |B| = 2n \geq |S|$.

8. Prove König's line coloring theorem: For every bipartite graph G, we have $\chi'(G) = \Delta(G)$.

   Hint: One proof is very similar (while simpler) to the proof of Vizing's theorem.

   Solution. We prove it by induction on $|E(G)|$. The theorem is obviously true for $|E(G)| = 1$. Now let us consider a bipartite graph G with $|E(G)| > 1$ and assume that the theorem is true for all bipartite graphs with $|E(G)| - 1$ edges. We use the following notations: for a given edge-coloring (i) $v \in V(G)$ is c-free if there is no edge incident at v colored with c, (ii) a $(c_1, \ldots, c_k)$-walk (resp. path, cycle) is a walk which contains only edges colored with $c_1, \ldots, c_k$.

   Let $xy \in E(G)$. Consider the graph $G - xy$: it is bipartite with $|E(G)| - 1$ edges and maximum degree at most $\Delta(G)$. By the induction assumption it has a proper $\Delta(G)$-edge-coloring, we call it C. x and y have degree at most $\Delta(G) - 1$ in $G - xy$, thus there exists colors c_x, c_y such that x is c_x-free and y is c_y-free in C. If $c_x = c_y$ then coloring xy with c_x extends C to a proper $\Delta(G)$-edge-coloring of G and we are done.

   Otherwise $c_x \neq c_y$. Consider $\mathcal{W}$ a maximal (c_x, c_y)-walk in $G - xy$ starting from x. Then no vertex can appear in $\mathcal{W}$ multiple times, otherwise there is more than one incident edge colored with c_x at the same vertex and C is not a proper edge-coloring. Hence $\mathcal{W}$ is indeed a path. Furthermore $y \notin \mathcal{W}$. If y is on $\mathcal{W}$, it is at the end of it, because y has no incident edge colored c_y. Then $\mathcal{W} + xy$ is an odd-length cycle, which is impossible in a bipartite graph.

   Then we can define a new edge-coloring C' that is identical to C for edges outside $\mathcal{W}$ and with swapped colors c_x, c_y on $\mathcal{W}$. Since $\mathcal{W}$ is a maximal (c_x, c_y)-path starting in x, C' is still a proper $\Delta(G)$-edge-coloring of $G - xy$. In C', x is now c_y-free while y is still c_y-free because $y \notin \mathcal{W}$. Therefore coloring xy with c_y extends C' to a proper $\Delta(G)$-edge-coloring of G.

9. Prove that every bipartite graph G has a matching of size at least $|E(G)|/\Delta(G)$.

   Solution. Consider an optimal edge coloring of G, which uses $\Delta(G)$ colors by König's line coloring theorem (proved above). Therefore, there are at least $|E(G)|/\Delta(G)$ edges colored by the same color, which form a matching.