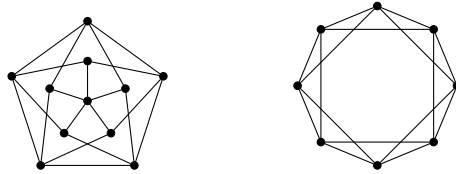


Graph Theory - Problem Set 6 (Solutions)

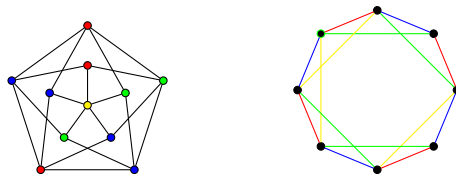
October 17, 2024

Exercises

1. Determine the chromatic number of the first graph and the edge-chromatic number of the second graph below.



Solution.



The chromatic number of the left graph and the edge-chromatic number of the right graph are both 4. Shown are 4-colorings for both.

To show that the coloring of the first graph is optimal, we try to 3-color it. Start with the outer C_5 : up to isomorphism there is only one coloring, red-blue-red-blue-green. This forces a red, blue, and green vertex on the inner ring of 5 vertices, which forces a fourth color on the middle vertex.

By Vizing's theorem we cannot color edges of the right graph by less than 4 colors.

2. For a graph G , we define $G[X]$, the subgraph *induced* by the vertex set $X \subseteq V(G)$ as the graph with vertex set X that contains all the edges of G with both ends in X . Prove that $\chi(G) \leq \chi(G[X]) + \chi(G[V \setminus X])$.

Solution. Define $\chi_1 = \chi(G[X])$, $\chi_2 = \chi(G[V \setminus X])$. We prove that there is a valid coloring of G with $\chi_1 + \chi_2$ colors: Color the vertices of X with χ_1 colors such that we have a valid coloring of $G[X]$, and color $V \setminus X$ with χ_2 colors different from the first χ_1 colors so that we get a valid coloring of $G[V \setminus X]$. Note that the edge e of G is either fully included in one of $G[X]$ or $G[V \setminus X]$, or it connects them. In both cases the end vertices of e get different colors: since in the former case both of the induced subgraphs have a valid coloring, and in the latter one, it follows from the fact that the vertices in X get different colors from the vertices in $V \setminus X$.

3. Are the following statements true? Provide reasons for your answers.

(a) If G and H are graphs on the same vertex set, then $\chi(G \cup H) \leq \chi(G) + \chi(H)$.

- (b) Every graph G has a coloring with $\chi(G)$ colors where $\alpha(G)$ vertices get the same color.

Solution. We give counterexamples to both of the statements:

- (a) Let $n \geq 6$ be a positive integer and V be a set of n vertices with the partition $V = V_1 \cup V_2$ such that $|V_1| = \lfloor \frac{n}{2} \rfloor$ and $|V_2| = \lceil \frac{n}{2} \rceil$. Let G, H be the graphs with the vertex set V satisfying

$$E(G) = \{vw : v, w \in V_1 \text{ or } v, w \in V_2\}, \quad E(H) = \{vw : v \in V_1, w \in V_2\}.$$

Note that G is the union of the complete graphs on V_1 and V_2 , so we have $\chi(G) \leq \max\{|V_1|, |V_2|\} = \lceil \frac{n}{2} \rceil$. On the other hand, H is a bipartite graph so $\chi(H) = 2$. Therefore, we have

$$\chi(G \cup H) = \chi(K_n) = n > \lceil \frac{n}{2} \rceil + 2 = \chi(G) + \chi(H).$$

- (b) Let $n > 1$ be a positive integer. Consider the *double star* graph G , i.e. G is the union of two disjoint star graphs $K_{1,n}$, such that the two centers of the stars are connected:

$$V = \{v_0, v_1, \dots, v_n, u_0, u_1, \dots, u_n\}, \quad E = \{v_0v_i : 1 \leq i \leq n\} \cup \{u_0u_i : 1 \leq i \leq n\} \cup \{v_0u_0\}.$$

It is easy to check that $\alpha(G) = |V(G)| - 2$, since the set of vertices $\{v_1, \dots, v_n, u_1, \dots, u_n\}$ is independent, and any independent set of vertices may have at most one of u_0 or u_i , for any i , and at most one of v_0 or v_i for any i . Furthermore, the set $\{v_1, \dots, v_n, u_1, \dots, u_n\}$ is the unique maximal independent set of vertices in G . On the other hand, we have $\chi(G) = 2$ since we can partition $V(G)$ into independent subsets $\{u_0, v_1, \dots, v_n\}, \{v_0, u_1, \dots, u_n\}$. Note that if we color all the vertices in $\{v_1, \dots, v_n, u_1, \dots, u_n\}$ with the same color, we need 3 colors to have a valid coloring of G . So this gives a counterexample to the statement.

Problems

4. Let G be a graph on n vertices and $\overline{G}$ be its complement. Prove that

- (a) $\chi(G)\chi(\overline{G}) \geq n$.
(b) $\chi(G) + \chi(\overline{G}) \leq n + 1$. *Hint: Prove by induction on n .*

Solution.

- (a) Note that the union $G \cup \overline{G}$ is the complete graph K_n . We construct a valid coloring of K_n with $\chi(G)\chi(\overline{G})$ colors. Then we get the required inequality, since $\chi(K_n) = n$. Denote $V = V(G)$. Let $c : V \rightarrow \{1, 2, \dots, \chi(G)\}$ be a valid coloring of G and $\bar{c} : V \rightarrow \{1, 2, \dots, \chi(\overline{G})\}$ be a valid coloring of $\overline{G}$. Define the coloring $c' : V \rightarrow \{1, 2, \dots, \chi(G)\} \times \{1, 2, \dots, \chi(\overline{G})\}$ with $c'(v) = (c(v), \bar{c}(v))$ for $v \in V$. It is easy to see that c' is a valid coloring for K_n on V : for distinct vertices $u, v \in V$, if $uv \in E(G)$, then $c(u) \neq c(v)$, and if $uv \in E(\overline{G})$, then $\bar{c}(u) \neq \bar{c}(v)$, both of which imply that $c'(u) \neq c'(v)$.
- (b) We prove it by induction on the number of the vertices. It is easy to check the induction basis. Now suppose the inequality holds for all graphs with n vertices, we prove it for the graph G on $n + 1$ vertices. Fix the vertex $v \in V(G)$ and let k be its degree in G , so the degree of v in $\overline{G}$ is $n - k$. Consider the graph $G - v$. Note that adding back

v to $G - v$ does not increase the chromatic number if $\chi(G - v) > k$, since one can color v by an existing color different from the colors of its k neighbors; otherwise, it will increase the chromatic number by at most one. The same statement holds for $\overline{G} - v$ with the condition $\chi(\overline{G} - v) > n - k$. Therefore, if at least one of $\chi(G - v) > k$ and $\chi(\overline{G} - v) > n - k$ holds, then applying induction hypothesis to $G - v$ will complete the proof:

$$\chi(G) + \chi(\overline{G}) \leq \chi(G - v) + \chi(\overline{G} - v) + 1 \leq n + 2.$$

Otherwise, we have $\chi(G - v) \leq k$ and $\chi(\overline{G} - v) \leq n - k$, which implies

$$\chi(G) + \chi(\overline{G}) \leq \chi(G - v) + \chi(\overline{G} - v) + 2 \leq k + n - k + 2 = n + 2.$$

This finishes the proof.

5. (a) Show that if an n -vertex graph is d -degenerate, then it has at most dn edges.
(b) Prove that if the longest path in G has length ℓ , then $\chi(G) \leq \ell + 1$.

Solution.

- (a) This can be proved by induction on the number of vertices n . Note that by the induction hypothesis, removing a vertex of degree at most d would result in a graph with at most $d(n - 1)$ edges, so the original graph has at most $d(n - 1) + d = dn$ edges.
 - (b) This follows from the fact that any such graph is ℓ -degenerate. To see this, let G' be a subgraph of G and v be an endpoint of a longest path in G' . Since this path cannot be extended, all the neighbors of v in G' are contained in this path, therefore $\deg_{G'}(v) \leq \ell$.
6. Let G be a 3-regular graph with $\chi'(G) = 4$. Prove that G does not have a Hamilton cycle.

Hint: What can be said about the parity of the number of vertices?

Solution. Note that since $\sum d(v) = 2|E(G)|$ and every $d(v) = 3$, G must have an even number of vertices. Suppose that G has a Hamilton cycle C . It must be even, so we can color it with 2 colors. Every vertex has 2 edges from C and one other edge, so the edges not in C form a matching. Hence we can color these edges with one color. This gives an edge-coloring of G with 3 colors, contradicting $\chi'(G) = 4$.

7. Prove that if every two odd cycles of G intersect in at least one vertex, then $\chi(G) \leq 5$.

Hint: Consider two cases in which G contains odd cycles or not.

Solution. If G has no odd cycles, then G is bipartite, which means that $\chi(G) \leq 2$. Thus we can assume that G has at least one odd cycle.

Let C be a shortest odd cycle, which is 3-colorable. Now remove the vertices of C from G to get a new graph $G - C$. It has no odd cycles, since every odd cycle previously intersected C . This implies that $G - C$ is bipartite, or in other words 2-colorable. Then we can combine a 2-coloring of $G - C$ with a 3-coloring of C to get a 5-coloring of G .

8. (a) Find the edge-chromatic number of K_{2n+1} .
(b) Find the edge-chromatic number of K_{2n} .

Solution.

(a) $\chi'(K_{2n+1}) = \Delta(K_{2n+1}) + 1 = 2n + 1.$

To get a $2n + 1$ edge-coloring, place the vertices v_i on a circle with equal spacing. Then for each vertex v_i , give the same color to the edges $v_{i-1}v_{i+1}$, $v_{i-2}v_{i+2}$, etc. (these edges will be parallel). This way we color all the edges with $2n + 1$ colors.

Suppose that we could color the edges of K_{2n+1} with $2n$ colors. Each color class has at most n edges, so with $2n$ colors we can color at most $2n^2$ edges. But K_{2n+1} has $\binom{2n+1}{2} = (2n + 1)n = 2n^2 + n$ edges, so this can't work.

(b) $\chi'(K_{2n}) = \Delta(K_{2n}) = 2n - 1.$

Now place $2n - 1$ of the vertices v_i on a circle with equal spacing, and put the remaining vertex u at the center of the circle. Then for each v_i , color in the same way as in the odd case, and also give that color to the edge uv_i . This gives an edge coloring with $2n - 1$ colors. In this case we have $\Delta(K_{2n}) = 2n - 1$, so by Vizing's theorem there is no edge coloring with fewer colors.